Sphärisch symmetrische Raumzeiten

Def. Eine Raumzeit (M,g) heißt sphärisch symmetrisch, falls die Gruppe $SO(3)$ auf (M,g) durch Isometrien wirkt deren Orbits raumartige 2-Sphären sind.

$\Rightarrow \exists$ Koordinaten $\{X^0, X^1, X^2, X^3\} = \{X^0, r, \theta, \varphi\}$ so daß

$$g = e^{2a}(dX^0)^2 - e^{2b}dr^2 - R^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{5A.1}$$

wobei a, b und R Funktionen von der Koordinaten $X^0 = ct$ und r sind.

Die $SO(3)$-Orbits sind 2-Sphären vom Flächeninhalt

$$A(X^0, r) = 4\pi R^2(X^0, r) \tag{5A.2}$$

Man nennt R den „Oberflächenradius".

Es ist

$$g = \theta^0 \otimes \theta^0 - \sum_{a=1}^{3} \theta^a \otimes \theta^a \tag{5A.3}$$

$$\left.\begin{aligned} \theta^0 &= e^a dX^0, & \theta^r &= e^b dr \\ \theta^\theta &= R\, d\theta, & \theta^\varphi &= R \sin\theta\, d\varphi \end{aligned}\right\} \tag{5A.4}$$

Wir nutzen zunächst die 1. Cartansche Strukturgleichung

$$d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = 0 \qquad (5A.5)$$

um die Zusammenhangskomponenten $\omega^\mu{}_\nu$ zu berechnen.

0: $d\theta^0 = e^a a' \, dr \wedge dx^0 = -e^{-b} a' \theta^0 \wedge \theta^r$

$= -\omega^0{}_\nu \wedge \theta^\nu$

$\Rightarrow \omega^0{}_r = e^{-b} a' \theta^0 + \sim \theta^r$

$$\omega^0{}_\theta \sim \theta^\theta \, , \quad \omega^0{}_\varphi \sim \theta^\varphi \qquad (5A.6)$$

r: $d\theta^r = e^b \dot{b} \, dx^0 \wedge dr = -e^{-a} \dot{b} \, \theta^r \wedge \theta^0$

$= -\omega^r{}_\nu \wedge \theta^\nu$

$\Rightarrow \omega^r{}_0 = e^{-a} \dot{b} \, \theta^r + \sim \theta^0$

$$\omega^r{}_\theta \sim \theta^\theta \, , \quad \omega^r{}_\varphi \sim \theta^\varphi \qquad (5A.7)$$

θ: $d\theta^\theta = \dot{R} \, dx^0 \wedge d\theta + R' \, dr \wedge d\theta$

$= -e^{-a} \frac{\dot{R}}{R} \theta^\theta \wedge \theta^0 - e^{-b} \frac{R'}{R} \theta^\theta \wedge \theta^r$

$= -\omega^\theta{}_\nu \wedge \theta^\nu$

$\Rightarrow \omega^\theta{}_0 = e^{-a} \frac{\dot{R}}{R} \theta^\theta + \sim \theta^0$

$\omega^\theta{}_r = e^{-b} \frac{R'}{R} \theta^\theta + \sim \theta^r$

$$\omega^\theta{}_\varphi \sim \theta^\varphi \qquad (5A.7)$$

$$\varphi: \quad d\Theta^\varphi = \dot R \sin\theta \, dx^0 \wedge d\varphi + R' \sin\theta \, dr \wedge d\varphi + R\cos\theta \, d\theta \wedge d\varphi$$

$$= -e^{-a}\frac{\dot R}{R}\,\Theta^\varphi \wedge \Theta^0 - e^{-b}\frac{R'}{R}\,\Theta^\varphi \wedge \Theta^r - \frac{\cot\theta}{R}\,\Theta^\varphi \wedge \Theta^\theta$$

$$= -\omega^\varphi{}_\nu \wedge \Theta^\nu$$

$$\Rightarrow \omega^\varphi{}_0 = e^{-a}\frac{\dot R}{R}\,\Theta^\varphi + \sim \Theta^0$$

$$\omega^\varphi{}_r = e^{-b}\frac{R'}{R}\,\Theta^\varphi + \sim \Theta^r$$

$$\omega^\varphi{}_\theta = \frac{\cot\theta}{R}\,\Theta^\varphi + \sim \Theta^\theta \qquad (5A.8)$$

Insgesamt ergibt sich für den Zusammenhang

$$\left.\begin{aligned}\omega^0{}_r = \omega^r{}_0 &= e^{-b} a' \,\Theta^0 + e^{-a}\dot b\,\Theta^r \\ &= e^{a-b} a' \, dx^0 + e^{b-a}\dot b\, dr\end{aligned}\right\} (5A.9a)$$

$$\omega^0{}_\theta = \omega^\theta{}_0 = e^{-a}\frac{\dot R}{R}\,\Theta^\theta = e^{-a}\dot R\, d\theta \qquad (5A.9b)$$

$$\omega^0{}_\varphi = \omega^\varphi{}_0 = e^{-a}\frac{\dot R}{R}\,\Theta^\varphi = e^{-a}\dot R \sin\theta\, d\varphi \qquad (5A.9c)$$

$$\omega^r{}_\theta = -\omega^\theta{}_r = -e^{-b}\frac{R'}{R}\,\Theta^\theta = -e^{-b} R'\, d\theta \qquad (5A.9d)$$

$$\omega^r{}_\varphi = -\omega^\varphi{}_r = -e^{-b}\frac{R'}{R}\,\Theta^\varphi = -e^{-b} R' \sin\theta\, d\varphi \qquad (5A.9e)$$

$$\omega^\theta{}_\varphi = -\omega^\varphi{}_\theta = -\frac{\cot\theta}{R}\,\Theta^\varphi = -\cos\theta\, d\varphi \qquad (5A.9f)$$

Damit haben wir durch die 1. Cart. St.-Gl. alle $\omega^\mu{}_\nu$ bestimmt.

Wir nutzen nun die 2. Cartanschen Struktur-gleichungen um den Krümmungstensor zu bestimmen:

$(0,r)$

$$\Omega^0{}_r = d\omega^0{}_r + \omega^0{}_\mu \wedge \omega^\mu{}_r$$

$$= d\omega^0{}_r + \cancel{\omega^0{}_\theta \wedge \omega^\theta{}_r} + \cancel{\omega^0{}_\varphi \wedge \omega^\varphi{}_r}$$

$$= d\omega^0{}_r$$

$$= \left\{ -\left(e^{a-b} a'\right)' + \left(e^{b-a} \dot b\right)^{\cdot} \right\} dx^0 \wedge dr$$

$$= \left\{ -e^{a-b}\left(a'' + a'(a'-b')\right) + e^{b-a}\left(\ddot b + \dot b(\dot b - \dot a)\right)\right\} dx^0 \wedge dr$$

$$= \left\{ e^{-2a}\left(\ddot b + \dot b(\dot b - \dot a)\right) - e^{-2b}\left(a'' + a'(a'-b')\right)\right\} \theta^0 \wedge \theta^r$$

$(0,\theta)$

$$\Omega^0{}_\theta = d\omega^0{}_\theta + \omega^0{}_r \wedge \omega^r{}_\theta + \cancel{\omega^0{}_\varphi \wedge \omega^\varphi{}_\theta}$$

$$= \left(e^{-a}\dot R\right)^{\cdot} dx^0 \wedge d\theta + \left(e^{-a}\dot R\right)' dr\, d\theta$$

$$+ \left(e^{-b} a' \theta^0 + e^{-a} \dot b \theta^r\right) \wedge \left(-e^{-b} \frac{R'}{R} \theta^\theta\right)$$

$$= \left[\frac{1}{R} e^{-a} \left(e^{-a} \dot R\right)^{\cdot} - e^{-2b} a' \frac{R'}{R}\right] \theta^0 \wedge \theta^\theta$$

$$+ \left[e^{-b} \frac{1}{R} \left(e^{-a} \dot R\right)' - e^{-a-b} \dot b \frac{R'}{R}\right] \theta^r \wedge \theta^\theta$$

$$= \left[\frac{1}{R} e^{-2a} \left(\ddot R - \dot a \dot R\right) - \frac{1}{R} e^{-2b} a' R'\right] \theta^0 \wedge \theta^\theta$$

$$+ \left[\frac{1}{R} e^{-a-b} \left(\dot R' - \dot R a'\right) - \frac{1}{R} e^{-a-b} \dot b R'\right] \theta^r \wedge \theta^\theta$$

$$\Omega^0{}_\theta = \frac{1}{R}\left[e^{-2a}(\ddot{R}-\dot{a}\dot{R}) - e^{-2b} a' R' \right] \theta^0 \wedge \theta^\theta$$

$$+ \frac{1}{R} e^{-(a+b)} \left[\dot{R}' - \dot{R} a' - \dot{b} R' \right] \theta^r \wedge \theta^\theta$$

$(0,\varphi)$ $\Omega^0{}_\varphi =$ (wegen $SO(3)$ symmetrie $\theta^\theta \leftrightarrow \theta^\varphi$)

$$= \frac{1}{R}\left[e^{-2a}(\ddot{R}-\dot{a}\dot{R}) - e^{-2b} a' R' \right] \theta^0 \wedge \theta^\varphi$$

$$+ \frac{1}{R} e^{-(a+b)} \left[\dot{R}' - \dot{R} a' - \dot{b} R' \right] \theta^r \wedge \theta^\theta$$

(r,θ) $\Omega^r{}_\theta = d\omega^r{}_\theta + \omega^r{}_0 \wedge \omega^0{}_\theta + \cancel{\omega^r{}_\varphi \wedge \omega^\varphi{}_\theta}$

$$= -(e^{-b} R')^{\cdot}\, dx^0 \wedge d\theta - (e^{-b} R')'\, dr \wedge d\theta$$

$$(e^{-b} a' \theta^0 + e^{-a} \dot{b} \theta^r) \wedge e^{-a} \frac{\dot{R}}{R} \theta^\theta$$

$$= \left[-\frac{1}{R} e^{-b} (e^{-b} R')' + \frac{1}{R} e^{-2a} \dot{b} \dot{R} \right] \theta^r \wedge \theta^\theta$$

$$+ \left[-\frac{1}{R} e^{-a} (e^{-b} R')^{\cdot} + \frac{1}{R} e^{-a-b} \dot{R} a' \right] \theta^0 \wedge \theta^\theta$$

$$= \frac{1}{R}\left[e^{-2a} \dot{b} \dot{R} - e^{-2b} (R'' - b' R') \right] \theta^r \wedge \theta^\theta$$

$$- \frac{1}{R} e^{-(a+b)} \left[\dot{R}' - R' \dot{b} - \dot{R} a' \right] \theta^0 \wedge \theta^\theta$$

(r,φ) $\quad \Omega^r{}_\varphi =$ (wegen SO(3) Symmetrie $\theta^\theta \leftrightarrow \theta^\varphi$)

$$= \frac{1}{R}\left[e^{-2a}\dot{b}\dot{R} - e^{-2b}(R'' - b'R')\right]\theta^r \wedge \theta^\varphi$$

$$- \frac{1}{R}e^{-(a+b)}\left[\dot{R}' - R'\dot{b} - \dot{R}a'\right]\theta^0 \wedge \theta^\varphi$$

(θ,φ) $\quad \Omega^\theta{}_\varphi = d\omega^\theta{}_\varphi + \omega^\theta{}_0 \wedge \omega^0{}_\varphi + \omega^\theta{}_r \wedge \omega^r{}_\varphi$

$$= \sin\theta \, d\theta \wedge d\varphi + \left(e^{-a}\frac{\dot{R}}{R}\right)^2 \theta^\theta \wedge \theta^\varphi - \left(e^{-b}\frac{R'}{R}\right)^2 \theta^\theta \wedge \theta^\varphi$$

$$= R^{-2}\left[1 + e^{-2a}\dot{R}^2 - e^{-2b}R'^2\right]\theta^\theta \wedge \theta^\varphi$$

Insgesamt ergibt sich für die Krümmung

$$R_{0r0r} = e^{-2a}(\ddot{b} + \dot{b}(\dot{b} - \dot{a})) - e^{-2b}(a'' + a'(a' - b')) \qquad (5A.10a)$$

$$R_{0\theta 0\theta} = \frac{1}{R}\left[e^{-2a}(\ddot{R} - \dot{a}\dot{R}) - e^{-2b}a'R'\right] = R_{0\varphi 0\varphi} \qquad (5A.10b)$$

$$R_{0\theta r\theta} = \frac{1}{R}e^{-(a+b)}\left[\dot{R}' - \dot{R}a' - \dot{b}R'\right] = R_{0\varphi r\varphi} \qquad (5A.10c)$$

$$R_{r\theta r\theta} = \frac{1}{R}\left[e^{-2b}(R'' - b'R') - e^{-2a}\dot{b}\dot{R}\right] = R_{r\varphi r\varphi} \qquad (5A.10d)$$

$$R_{r\theta 0\theta} = \frac{1}{R}e^{-(a+b)}\left[\dot{R}' - R'\dot{b} - \dot{R}a'\right] = R_{r\varphi 0\varphi} \qquad (5A.10e)$$

$$R_{\theta\varphi\theta\varphi} = -\frac{1}{R^2}\left[1 + e^{-2a}\dot{R}^2 - e^{-2b}R'^2\right] \qquad (5A.10f)$$

Für den Ricci-Tensor

$$R_{00} = -R_{0r0r} - R_{0\theta0\theta} - R_{0\varphi0\varphi}$$

$$= -R_{0r0r} - 2R_{0\theta0\theta}$$

$$= -e^{-2a}\left(\ddot{b} + \dot{b}^2 - \dot{a}\dot{b} + \frac{2\ddot{R}}{R} - \dot{a}\frac{2\dot{R}}{R}\right)$$

$$+ e^{-2b}\left(a'' + a'^2 - a'b' + 2a'\frac{R'}{R}\right) \qquad (5A.11a)$$

$$R_{0r} = -R_{0\theta r\theta} - R_{0\varphi r\varphi} = -2R_{0\theta r\theta}$$

$$= -\frac{2}{R}e^{-(a+b)}\left[\dot{R}' - R'\dot{b} - \dot{R}a'\right] \qquad (5A.11b)$$

$$R_{0\theta} = -R_{0r\theta r} - R_{0\varphi\theta\varphi} = R_{0\varphi} = 0 \qquad (5A.11c)$$

$$R_{rr} = R_{r0r0} - 2R_{r\theta r\theta}$$

$$= e^{-2a}\left(\ddot{b} + \dot{b}^2 - \dot{a}\dot{b} + 2\dot{b}\dot{R}/R\right)$$

$$- e^{-2b}\left(a'' + a'^2 - a'b' + 2R''/R - 2R'b'/R\right) \qquad (5A.11d)$$

$$R_{r\theta} = R_{r\varphi} = R_{r0\theta0} + R_{r\varphi\theta\varphi} = 0 \qquad (5A.11e)$$

$$R_{\theta\theta} = R_{\varphi\varphi} = R_{\theta0\theta0} - R_{\theta r\theta r} - R_{\theta\varphi\theta\varphi}$$

$$= e^{-2a}\left(\frac{1}{R}(\ddot{R} - \dot{a}\dot{R}) + \dot{b}\dot{R}/R\right)$$

$$+ e^{-2b}\left(-a'R'/R - (R'' - b'R')/R\right)$$

$$+ \frac{1}{R^2}\left(1 + e^{-2a}\dot{R}^2 - e^{-2b}R'^2\right)$$

$$
\begin{aligned}
R_{\theta\theta} &= R_{\varphi\varphi} \\
&= e^{-2a}\left[\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{\dot{R}}{R}(\dot{b}-\dot{a})\right] \\
&\quad - e^{-2b}\left[\frac{R''}{R} + \frac{R'^2}{R^2} + \frac{R'}{R}(a'-b')\right] \\
&\quad + \frac{1}{R^2}
\end{aligned}
\qquad (5A.11f)
$$

$$
R_{\theta\varphi} = R_{\theta 0 \varphi 0} - R_{\theta r \varphi r} = 0. \qquad (5A.11g)
$$

Ricci-Skalar

$$
\begin{aligned}
R &= R_{00} - R_{rr} - R_{\theta\theta} - R_{\varphi\varphi} \\
&= -(-R_{0r0r} - 2R_{0\theta 0\theta}) \\
&\quad - (R_{r0r0} - 2R_{r\theta r\theta}) \\
&\quad - 2(R_{\theta 0\theta 0} - R_{\theta r\theta r} - R_{\theta\varphi\theta\varphi}) \\
&= -2R_{0r0r} - 4R_{0\theta 0\theta} + 4R_{r\theta r\theta} + 2R_{\theta\varphi\theta\varphi} \\
&= 2(R_{\theta\varphi\theta\varphi} - R_{0r0r}) + 4(R_{r\theta r\theta} - R_{0\theta 0\theta})
\end{aligned}
\qquad (5A.12)
$$

Einstein Tensor

$$G_{00} = R_{00} - \frac{1}{2}R = \frac{1}{2}\left(R_{00} + R_{rr} + 2R_{\theta\theta}\right)$$

$$= \left(-\cancel{R_{0r0r}} - 2\cancel{R_{0\theta0\theta}}\right)$$

$$- R_{\theta\varphi\theta\varphi} + \cancel{R_{0r0r}} - 2R_{r\theta r\theta} + 2\cancel{R_{0\theta0\theta}}$$

$$= -\left(R_{r\theta r\theta} + R_{r\varphi r\varphi} + R_{\theta\varphi\theta\varphi}\right)$$

$$= -\frac{2}{R}\left[e^{-2b}(R'' - b'R') - e^{-2a}\dot{b}\dot{R}\right]$$

$$+ \frac{1}{R^2}\left[1 + e^{-2a}\dot{R}^2 - e^{-2b}R'^2\right] \qquad (5A.13a)$$

$$G_{0r} = R_{0r} = -R_{0\theta r\theta} - R_{0\varphi r\varphi} = -2R_{0\theta r\theta}$$

$$= -\frac{2}{R}e^{-(a+b)}\left[\dot{R}' - \dot{R}a' - \dot{b}R'\right] \qquad (5A.13b)$$

$$G_{0\theta} = R_{0\theta} = 0 \qquad (5A.13c)$$

$$G_{0\varphi} = R_{0\varphi} = 0 \qquad (5A.13d)$$

$$G_{rr} = R_{rr} + \frac{1}{2}R = \frac{1}{2}\left(R_{00} + R_{rr} - 2R_{\theta\theta}\right)$$

$$= \cancel{R_{r0r0}} - 2\cancel{R_{r\theta r\theta}}$$

$$+ R_{\theta\varphi\theta\varphi} - \cancel{R_{0r0r}} + 2\cancel{R_{r\theta r\theta}} - 2R_{0\theta0\theta}$$

$$= R_{\theta\varphi\theta\varphi} - R_{0\theta0\theta} - R_{0\varphi0\varphi}$$

$$= -\frac{2}{R}\left[e^{-2a}(\ddot{R} - \dot{a}\dot{R}) - e^{-2b}a'R'\right]$$

$$- \frac{1}{R^2}\left[1 + e^{-2a}\dot{R}^2 - e^{-2b}R'^2\right] \qquad (5A.14e)$$

$$G_{r\theta} = R_{r\theta} = 0 \qquad (5A.14f)$$

$$G_{r\varphi} = R_{r\varphi} = 0 \qquad (5A.14g)$$

$$G_{\theta\theta} = R_{\theta\theta} + \tfrac{1}{2} R = \tfrac{1}{2}(R_{00} - R_{rr})$$

$$= \cancel{R_{\theta 0\theta 0}} - \cancel{R_{\theta r\theta r}} - \cancel{R_{\theta\varphi\theta\varphi}}$$

$$+ \cancel{R_{\theta\varphi\theta\varphi}} - R_{0r0r} + \cancel{2} R_{r\theta r\theta} - \cancel{2} R_{0\theta 0\theta}$$

$$= R_{r\theta r\theta} - R_{0r0r} - R_{0\theta 0\theta}$$

$$= R_{r\varphi r\varphi} - R_{0\varphi 0\varphi} - R_{0r0r}$$

$$= - e^{-2a}\left[\ddot{b} + \dot{b}^2 - \dot{a}\dot{b} + \frac{\ddot{R}}{R} - (\dot{a} - \dot{b})\frac{\dot{R}}{R}\right]$$

$$+ e^{-2b}\left[a'' + a'^2 - a'b' + \frac{R''}{R} + (a' - b')\frac{R'}{R}\right] \qquad (5A.14h)$$

$$G_{\varphi\varphi} = G_{\theta\theta} \qquad (5A.14i)$$

$$G_{\theta\varphi} = R_{\theta\varphi} = 0. \qquad (5A.14j)$$

Spezialisierungen

1. Fall $R = r \curvearrowright \dot{R} = 0,\ R' = 1,\ R'' = 0$

$$G_{00} = \frac{1}{r^2}\left[1 - e^{-2b}\right] + \frac{2b'}{r} e^{-2b}$$

$$= \frac{1}{r^2}\left[r\left(1 - e^{-2b}\right)\right]' \qquad (5A.15a)$$

$$G_{0r} = \frac{2\dot{b}}{r} e^{-(a+b)} \qquad (5A.15b)$$

$$G_{0\theta} = G_{0\varphi} = 0 \qquad (5A.15c)$$

$$G_{rr} = -\frac{1}{r^2}\left[1 - e^{-2b}\right] + \frac{2a'}{r} e^{-2b} \qquad (5A.15d)$$

$$G_{r\theta} = G_{r\varphi} = 0 \qquad (5A.15e)$$

$$G_{\theta\theta} = G_{\varphi\varphi} =$$

$$= -e^{-2a}\left[\ddot{b} + \dot{b}^2 - \dot{a}\dot{b}\right]$$

$$+ e^{-2b}\left[a'' + a'^2 - a'b' + \frac{1}{r}(a' - b')\right]$$

$$(5A.15f)$$

$$G_{\theta\varphi} = 0. \qquad (5A.15g)$$

Beachte: Alle Komponenten beziehen sich hier auf die orthonormierte Basis der $\{\theta^0, \dots, \theta^3\}$.

Bezogen auf die dx^μ und mit

$\dot{} = \frac{\partial}{\partial t}$ statt $\frac{\partial}{\partial x^0}$ ist

$$G_{00} = \frac{e^{2a}}{r^2}\left[r\left(1-e^{-2b}\right)\right]' \qquad (5A.16a)$$

$$G_{0r} = \frac{2\dot{b}}{cr} \qquad (5A.16b)$$

$$G_{rr} = -\frac{1}{r^2}\left[e^{2b}-1\right] + \frac{2a'}{r} \qquad (5A.16c)$$

$$G_{\theta\theta} = r^2\Big\{-e^{-2a}\left[\ddot{b}+\dot{b}^2-\dot{a}\dot{b}\right] + e^{-2b}\left[a''+a'^2-a'b'+\frac{1}{r}(a'-b')\right]\Big\} \qquad (5A.16d)$$

$$G_{\varphi\varphi} = \sin^2\theta\, G_{\theta\theta} \qquad (5A.16e)$$