Special Topics in GR

(WS 2020/21)

We start these lectures with an introduction into Cosmology – eventually relativistic, but first Newtonian in order to set the stage.

Lecture 1: Newtonian Cosmology

- On the largest observable scales, i.e., $\sim 10^{10}$ ly (light years), the visible matter seems to be distributed homogeneously – more or less !

- Typical structures ("particles") from the 10^{10} ly – point of view are galaxies, whose diameters are typically

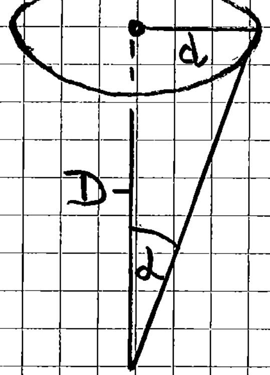
$$D \cong 10^5 \text{ ly} \tag{1.1}$$

Note

$$\text{ly} = 299\,792\,458 \cdot \frac{m}{s} \cdot 3600 \cdot 24 \cdot 365 \text{ s}$$

$$= 9.45 \times 10^{15} \text{ m} \cong 10^{16} \text{ m} \tag{1.2}$$

- Typical distances (between relevant structures) are measured in units of Parsec („paralactic second"). It is defined as that distance from which the Earth-Sun-radius appears to span an arc second (1/3600 of a degree):



$$d = 150 \cdot 10^6 \text{ km} = 1.5 \cdot 10^8 \text{ km}$$

$$\alpha = 1'' = 1^\circ / 3600$$
$$= \frac{2\pi}{360 \cdot 3600} \text{ rad} \qquad (1.3)$$

$$\frac{d}{D} = \tan\alpha \cong \alpha$$

$$\Rightarrow \quad D = 1\,\text{pc} = \frac{d}{\alpha} = \frac{1.5 \cdot 10^8 \text{ km}}{2\pi / 360 \cdot 3600}$$
$$= 3{,}1 \times 10^{13} \text{ km}$$
$$= 3{,}26 \text{ ly} \qquad (1.4)$$

- The diameter of a spiral galaxy like ours (Milky Way) or Andromeda is typically of the order of

$$10^5 \text{ ly} = 3 \times 10^4 \text{ pc} \qquad (1.5)$$

Hence the diameter of visible universe is

$$10^{10} \text{ ly} \cong 3 \times 10^5 \text{ galaxy diam.} \qquad (1.6)$$

- Distance to next galaxy (Andromeda) $\sim 2 \cdot 10^6$ ly $\Rightarrow$ opening angle

$$\alpha = \frac{\text{size}}{\text{distance}} = \frac{10^5}{2 \cdot 10^6} = \frac{1}{20} \text{ rad} \qquad (1.7)$$

correspondending to

$$\frac{360}{2\pi} \frac{1}{20} \text{ deg} = 2.86° \cong 3° \qquad (1.8)$$

i.e. 6 x full moon.

- <u>Galaxy clusters</u> are sets of $n \cdot 10^2 - n \cdot 10^3$ galaxies in a gravity-bound state (i.e. held in a compact region of space by their mutual gravitational attraction). There are about $10^{11} - 2 \cdot 10^{11}$

Stars in the Milky Way with average mass 10^{31} kg $\Rightarrow M_{MW} \cong 10^{42}$ kg.
Hence galaxy clusters combine total masses of about $10^{14} - 10^{16} M_\odot$

$$M_\odot = 2 \cdot 10^{30} \text{ kg} \tag{1.9}$$

$$= \text{Solar mass}$$

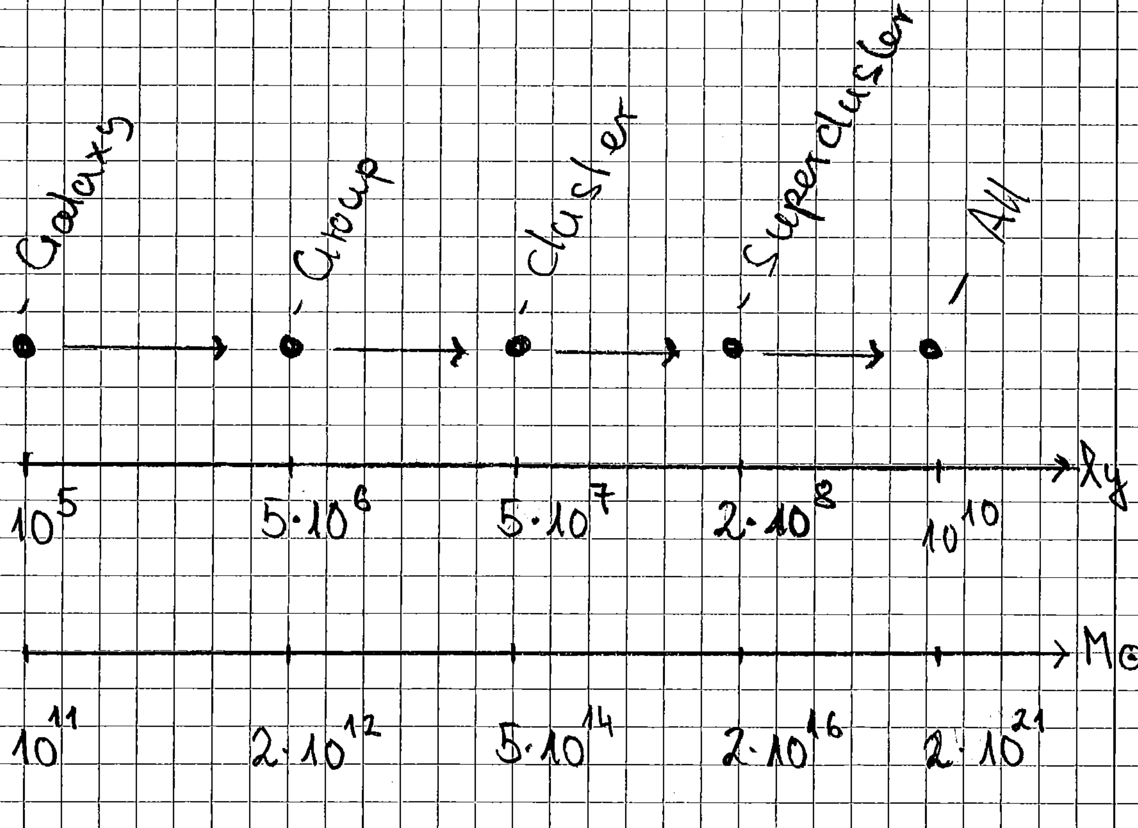
in $10^2 - 10^3$ stars of average mutual distance $n \cdot 10^6$ ly $\cong$ Mpc.

Example: Virgo cluster, of which we are part, with centre ~ 16 Mpc in the direction of constellation Virgo, comprising about 1300-2000 galaxies (containing MW) of total mass $10^{15} M_\odot$ within diameter of 2 Mpc. Another member galaxy is M87 (Virgo A), which is a supergiant elliptical galaxy of about 10^{12} stars and a supermassive Black Hole of

$$M_{BH}^{(M87)} = (6.4 \pm 0.5) \times 10^9 M_\odot \tag{1.10}$$

[The BH in NGC 4889 has about 3 times that mass!]

- **Galaxy superclusters** are the largest structures in the visible universe, combining $10^2 - 10^3$ clusters, i.e. $\sim 10^5$ galaxies with total $10^{16}-10^{17}$ solar masses, on a length scale of up to 10-100 Mpc ($\sim 10^8$ ly).
  The important fact about superclusters is, that they are the first structures from and above which no gravitational binding takes place. That is, superclusters take part in the cosmological expansion (Hubble flow).



(1.1)

Of the total mass of the visible Universe, only 4-5 % seems to be of known (baryonic) nature ! (→ missing mass)

Lets take a naive point of view and model the dynamics - on average and on largest scales ($\sim 10^{10}$ ly) by a "gas" of pointlike objects (galaxies) which are described like a perfect fluid by

$$
\left.
\begin{array}{lll}
\text{mass density} & : & \varrho(t,\vec{x}) \\
\text{pressure} & : & p(t,\vec{x}) \\
\text{velocity} & : & \vec{v}(t,\vec{x})
\end{array}
\right\} \quad (1.12)
$$

Obeying the following laws:

1.) conservation of mass

$$\dot{\varrho} + \vec{\nabla}(\varrho\vec{v}) = 0 \qquad (1.13)$$

$\rightsquigarrow$ mass in volume $\Omega \subset \mathbb{R}^3$ at time t

$$M(\Omega,t) = \int_\Omega \varrho(t,\vec{x})\, d^3x$$

$$
\begin{aligned}
\frac{d}{dt} M(\Omega,t) &= \int_\Omega \dot{\varrho}(t,\vec{x})\, d^3x \\
&= -\int_\Omega \vec{\nabla}(\varrho\vec{v})\, d^3x \\
&= \int_{\partial\Omega} -\varrho(\vec{v}\cdot\vec{n})\, d\sigma \qquad (1.14)
\end{aligned}
$$

$$\Rightarrow \quad -\frac{dM(\Omega,t)}{dt} = \int_{\partial\Omega} (\varrho\vec{v})\cdot\vec{n}\, d\sigma \qquad (1.15)$$

The time-rate of diminishment of mass in volume Ω equals the time-rate of outflux through boundary $\partial\Omega$ of Ω.

2.) Gravitational interaction of masses via

Gravitational potential ϕ obeying

$$\Delta\phi = 4\pi G \varrho \qquad (1.16)$$

leading to force density

$$\vec{f} = -\varrho\vec{\nabla}\phi \qquad (1.17)$$

acting upon masses which then move according to Newtonian law of motion (if pressure $p=0$)

$$\underset{\substack{\uparrow\\ \text{inertial mass}}}{\varrho}\dot{\vec{v}} = \vec{f} = -\underset{\substack{\uparrow\\ \text{gravitational mass}}}{\varrho}\vec{\nabla}\phi . \qquad (1.18)$$

For non-zero pressure we start in a field-theoretic way from momentum conservation

Momentum density $\vec{p} = \varrho \vec{v}$

conservation of momentum

$$\partial_t (\varrho v^a) + \partial_b (\varrho v^a v^b)$$

$$= - \partial^a p - \varrho \partial^a \phi \qquad (1.19)$$

change of mom. due to pressure-gradient

change of mom. due to grav. force

Using (1.13), i.e.

$$\partial_a (\varrho v^a) = - \dot{\varrho}$$

the lhs of (1.19) reads

$$\cancel{\dot{\varrho} v^a} + \varrho \dot{v}^a + v^a \cancel{\partial_b (\varrho v^b)} + \varrho v^b \partial_b v^a$$

$$= \varrho \underbrace{(\dot{v}^a + v^b \partial_b v^a)}_{\text{"co-moving time derivative"}} \qquad (1.20)$$

$$= \text{"} \varrho \frac{D}{Dt} v^a \text{"}$$

Hence the "cosmological fluid" obeys Euler-Equation

$$\dot{V}^a + V^b \partial_b V^a = -\frac{\partial^a p}{\varrho} - \partial^a \Phi \qquad (1.21)$$

A co-moving (or cosmological) observer is an observer moving with velocity field $\vec{V}(t,\vec{x})$.

The observer at spatial position $\vec{a}$ measures relative to his/her rest-frame a velocity field

$$\vec{W}_{\vec{a}}(t,\vec{x}) := \vec{V}(t,\vec{x}+\vec{a}) - \vec{V}(t,\vec{a}) \qquad (1.22)$$

We now invoke the

Cosmological or Copernican Principle:

No co-moving observer is distinguished. In other words: All co-moving observers see the same matter configuration

$$\Rightarrow \quad \left.\begin{aligned} \varrho(t,\vec{x}) &= \varrho(t) \\ p(t,\vec{x}) &= p(t) \end{aligned}\right\} \text{ spatially constant} \qquad (1.23)$$

$$\vec{W}_{\vec{a}}(t,\vec{x}) = \text{independent of } \vec{a} \qquad (1.24)$$

The last condition implies the following:

$$\vec{v}(t, \vec{x} + \vec{a}) - \vec{v}(t, \vec{a})$$
$$= \vec{v}(t, \vec{x}) - \vec{v}(t, \vec{0}) \qquad (1.25)$$

Set

$$\vec{h}(t, \vec{x}) := \vec{v}(t, \vec{x}) - \vec{v}(t, \vec{0}) \qquad (1.26)$$

then (1.25) reads

$$\vec{h}(t, \vec{x} + \vec{a}) - \vec{h}(t, \vec{a}) = \vec{h}(t, \vec{x})$$

$$\Leftrightarrow \quad \vec{h}(t, \vec{x} + \vec{a}) = \vec{h}(t, \vec{x}) + \vec{h}(t, \vec{a})$$

$\Leftrightarrow \quad \vec{h}(t, \cdot)$ is linear in 2nd argument $= A(t) \cdot \vec{x}$

$$\Leftrightarrow \quad v(t, \vec{x}) = A(t) \cdot \vec{x} + \vec{v}_0(t) \qquad (1.27)$$

where $A(t)$ is a one-parameter family of linear maps $\mathbb{R}^3 \to \mathbb{R}^3$, i.e. a curve (differentiable) in $\mathrm{End}(\mathbb{R}^3)$.

So far we have not put any constraints on the coordinate system with respect to which our components of $\vec{v}$ refer.

We now restrict that system so that

$$\vec{V}_0(t) = 0 \qquad (1.28)$$

$$A(t) = A^T(t) \qquad (1.29)$$

The first just means that the origin, $\vec{X}=\vec{0}$, is co-moving, i.e.

$$\vec{V}(t, \vec{X}=0) = \vec{0} \quad \forall t. \qquad (1.30)$$

The second, (1.29), means that there is no overall rotation.

Lemma: We can always achieve (1.28) and (1.29) by a time dependent Euklidean motion

$$\vec{X} \mapsto \vec{X}' := R(t)\vec{X} + \vec{b}(t) \qquad (1.31)$$

where $R(t) \in SO(3)$

Proof. We first derive how $\vec{V}(t, \vec{X})$ changes under (1.31). To see that, let $\vec{X}(t)$ be the flow for $\vec{V}$, i.e.

$$\dot{\vec{X}}(t) = V(t, \vec{X}(t)).$$

The (1.31)-transformed flow is

$$\vec{X}'(t) = R(t)\vec{X}(t) + b(t) \tag{1.32}$$

Its corresponding vector field is (we suppress the arguments t)

$$\begin{aligned}
\dot{\vec{X}}' &= R\vec{V} + \dot{R}\vec{X} + \dot{\vec{b}} \\
&= R(A\vec{X} + \vec{V}_0) + \dot{R}\vec{X} + \dot{\vec{b}} \\
&= R(A + R^T\dot{R})\vec{X} + R\vec{V}_0 + \dot{\vec{b}} \\
&= R(A + R^T\dot{R})R^T(\vec{X}' - \vec{b}) + R\vec{V}_0 + \dot{\vec{b}} \\
&= \Theta'\vec{X}' + (\dot{\vec{b}} + R\vec{V}_0 - \Theta'\vec{b})
\end{aligned} \tag{1.33}$$

where

$$\Theta'(t) = R(t)\left(A(t) + R^T(t)\dot{R}(t)\right)R^T(t) \tag{1.34}$$

We can choose $R(t)$ such that Θ' is symmetric, i.e. that the anti-symmetric part of Θ' vanishes. Indeed, $R^T\dot{R}$ is already antisymmetric,

$$\begin{aligned}
(R^T\dot{R})^T &= \dot{R}^T R = (\underbrace{R^T R}_{=id})^{\cdot} - R^T\dot{R} \\
&= -R^T\dot{R}
\end{aligned} \tag{1.35}$$

Let

$$\hat{A} = \tfrac{1}{2}\left(A - A^T\right) \tag{1.36}$$

denote the antisymmetric part of A, we can choose $R(t)$ according to

$$\hat{A} + R^T \dot{R} = 0 \tag{1.37}$$

$$\Leftrightarrow \quad \dot{R}(t) = -R(t)\,\hat{A}(t) \tag{1.38}$$

which is a linear ODE for $R(t)$ that can always be solved. Then

$$\Theta'(t) = R(t)\,\tilde{A}(t)\,R^T(t) \tag{1.39}$$

where

$$\tilde{A} = \tfrac{1}{2}\left(A + A^T\right) \tag{1.40}$$

is the symmetric part. Obviously

$$(\Theta')^T = \Theta' \tag{1.41}$$

Finally we choose $\vec{b}(t)$ according to

$$\dot{\vec{b}}(t) = \Theta'(t)\,\vec{b}(t) - R(t)\,\vec{v}_0(t) \tag{1.42}$$

which is again a linear inhomog. ODE.

Then

$$\dot{\vec{X}}'(t) = \Theta'(t)\,\vec{X}'(t) = \vec{V}'(t, \vec{X}'(t)) \qquad (1.43)$$

or $$\vec{V}'(t, \vec{X}') = \Theta'(t)\,\vec{X}' \qquad (1.44)$$

with symmetric Θ'. $\blacksquare$

Let now

$$V(t, \vec{X}) = A(t)\,\vec{X} \qquad (1.45)$$

with $A(t) = A^T(t)$ $\qquad (1.46)$

and $\varrho = \varrho(t)$ $\qquad (1.47)$

and $p \equiv 0$ (w.l.o.g since only $\vec{\nabla} p$ enters) $\qquad (1.48)$

From mass conservation

$$\dot{\varrho} + \vec{\nabla}(\varrho \vec{V}) = 0 \qquad (1.13)$$

get $$\frac{\dot{\varrho}}{\varrho} = -\mathrm{Tr}(A) \qquad (1.49)$$

and momentum conservation

$$\dot{V}^a + V^b \partial_b V^a = -\partial^a \phi \qquad (1.21)$$

$$\Leftrightarrow \quad \dot{A}^a{}_b X^b + A^a{}_b A^b{}_c X^c = -\partial^a \phi \qquad (1.50)$$

Taking the divergence of (1.49), i.e., applying ∂a to it, we get

$$\mathrm{Tr}(\dot{A}) + \mathrm{Tr}(A^2) = -\Delta\phi$$

$$\underset{\substack{\uparrow \\ \text{using (1.16)}}}{=} -4\pi G \rho \qquad (1.51)$$

Note that

$$\mathrm{Tr}(A^2) = \mathrm{Tr}(A A^T) \geqslant 0 \qquad (1.52)$$

with equality $\Leftrightarrow$ $A \equiv 0$. Since $\rho \geqslant 0$ (1.50) shows

$$\mathrm{Tr}(\dot{A}) < 0 \text{ for } \rho \neq 0 \qquad (1.53)$$

Another such inequality arises as follows: Take the time-derivative ∂t of (1.49)

$$\partial t \left(\frac{\dot{\rho}}{\rho}\right) = [\log(\rho)]^{\cdot\cdot} = -\mathrm{Tr}(\dot{A}) \qquad (1.54)$$

Then, with (1.51),

$$[\log(\rho)]^{\cdot\cdot} = \underbrace{\mathrm{Tr}(A^2)}_{\geqslant 0} + \underbrace{4\pi G \rho}_{\geqslant 0} \geqslant 0 \qquad (1.55)$$

and $= 0 \Leftrightarrow A \equiv 0$ and $\rho = 0$

This shows that ρ is time independent iff $\rho \equiv 0$. Hence we have

<u>Theorem</u>: A non-empty Newtonian universe obeying the copernican Principle cannot be static

This seems surprising, since one might have anticipated a static homogeneous (i.e. constant) mass distribution to be a solution. However, if $\vec{V} = 0$ then $A = 0$ then $\partial^a \phi = 0$ (by, e.g., (1.507)), then $\Delta \phi = 0$ and therefore $\rho = 0$ by the field equation (1.16)

There is still a puzzle, though! Our equations require $\vec{V} \neq 0$ for $\partial^a \phi \neq 0$. But what direction should $\vec{V}$ point to if all the matter is distributed homogeneously?

Can you resolve that puzzle? What is wrong with $\rho = \text{const.} > 0$?

The cosmological constant in Newtonian cosmology

In order to get static solutions one has to modify the field equations for the gravitational field. This has been suggested long before Einstein in a Newtonian context by Carl Neumann and Hugo Seeliger.
The idea is simple: set

$$\Delta \phi + \Lambda = 4\pi G \rho \tag{1.56}$$

i.e.
$$\Delta \phi = 4\pi G (\rho - \bar{\rho}) \tag{1.57}$$

where
$$\bar{\rho} := \Lambda / 4\pi G, \quad \Lambda = \text{const.} \tag{1.58}$$

Then, instead of (1.55) get

$$[\log(\rho)]^{\cdot\cdot} = \mathrm{Tr}(A^2) + 4\pi G(\rho - \bar{\rho}) \tag{1.59}$$

and
$$\left.\begin{aligned} \rho = \bar{\rho} &> 0 \\ A &\equiv 0 \end{aligned}\right\} \tag{1.60}$$

is a solution

This Λ-Term is the Newtonian analog to the Λ-Term in GR. However, the analogy is not perfect, for here a positive Λ corresponds to constant negative effective background mass density $-\bar{\rho}$, whereas in GR a positive Λ corresponds to a positive mass/energy density. The reason why in GR this still corresponds to an effective repulsion, thereby stabilizing the gravitational attraction, is that in GR it is not ρ but rather the combination

$$(\rho + 3\,p/c^2) \tag{1.61}$$

that drives the gravitational potential (to speak in Newtonian terms). Moreover, in GR a positive Λ corresponds to

$$\rho_\Lambda = \Lambda / 8\pi G \tag{1.62a}$$

$$p_\Lambda = -\rho_\Lambda c^2 \tag{1.62b}$$

so that

$$(\rho_\Lambda + 3\,p_\Lambda/c^2) = -2\,\rho_\Lambda = -\frac{\Lambda}{4\pi G} \tag{1.63}$$

So it is Λ's contribution to pressure rather than energy/mass density that is responsible for repulsion. Due to the appearance of c^2 in (1.61) there is clearly no Newtonian analog to pressure contributing to ϕ's source.

Note that the radially symmetric solution of (1.56) for a point source follows from

$$\Delta \phi = 4\pi G M \delta^{(3)}_{\vec{0}} - \Lambda$$

$$= \phi'' + \frac{2}{r}\phi' = \frac{1}{r}(r\phi)'' \qquad (1.64)$$

$$\Rightarrow \quad \phi(r) = -G\frac{M}{r} - \frac{\Lambda}{6}r^2 \qquad (1.65)$$

$$\rightsquigarrow \quad \vec{g}(\vec{x}) = -\vec{\nabla}\phi(\vec{x}) = -G\frac{M}{r^3}\vec{x} + \frac{\Lambda}{3}\vec{x} \qquad (1.66)$$

linearly growing repulsive force.