Lecture 11 : Λ - dominated Universes

Def. A FLRW Universe is called Λ - dominated, if $\rho = p = 0$, i.e. only Ω_k and Ω_Λ are non-zero.

For $\rho = p = 0$, Friedmann's equations (6.16 a-c) reduce to the middle one [(6.16c) is trivially satisfied, $0 \equiv 0$, and (6.16a) follows from differentiating (6.16b).]. Hence Friedmann's equations are equivalent to (6.53) with

$$\left(\frac{dx}{d\lambda}\right)^2 - \Omega_\Lambda x^2 = \Omega_k \tag{11.1}$$

or, setting $\Omega_\Lambda = \Omega$
and using $\Omega_k = 1 - \Omega$, } (11.2)

$$\left(\frac{dx}{d\lambda}\right)^2 - \Omega x^2 = (1-\Omega) \tag{11.3}$$

where

$$\lambda := H_0 t \tag{11.4}$$

$$x := a/a_0 \tag{11.5}$$

$$\Omega = \Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} \tag{11.6}$$

$$\Omega_k = 1 - \Omega = -kc^2/(H_0 a_0)^2 \tag{11.7}$$

We have to distinguish the following cases

$$\left.\begin{array}{l} \text{1. Case: } \Omega > 1 \\ \iff \Omega_k < 0 \iff k = +1 \end{array}\right\} \quad (11.8)$$

$$\left.\begin{array}{l} \text{2. Case } \Omega = 1 \\ \Rightarrow \Omega_k = 0,\ k = 0 \end{array}\right\} \quad (11.9)$$

$$\left.\begin{array}{l} \text{3. case } 0 < \Omega < 1 \\ \Rightarrow \Omega_k > 0,\ k = -1 \end{array}\right\} \quad (11.10)$$

$$\left.\begin{array}{l} \text{4. case } \Omega = 0 \\ \iff \Omega_k = 1,\ k = -1 \end{array}\right\} \quad (11.11)$$

$$\left.\begin{array}{l} \text{5. case } \Omega < 0 \\ \iff \Omega_k > 1,\ k = -1 \end{array}\right\} \quad (11.12)$$

Note that the 4. case, $\Omega = 0$, is not really in our class of Ω_Λ "domination" but we include it here anyway for reasons of systematics and completeness.

We now integrate the Friedmann-eqn. (11.1) for each case in turn.

Case 1

$$\left(\frac{dx}{d\lambda}\right)^2 = \Omega x^2 - \underbrace{(\Omega - 1)}_{>0} \qquad (11.13)$$

$$\frac{dx}{\left[\Omega x^2 - (\Omega - 1)\right]^{1/2}} = \pm\, d\lambda$$

$$\frac{1}{(\Omega - 1)^{1/2}} \frac{dx}{\left\{\left(\sqrt{\frac{\Omega}{\Omega-1}}\, x\right)^2 - 1\right\}^{1/2}} = \pm\, d\lambda$$

$$\frac{1}{\sqrt{\Omega}} \frac{dz}{\left(z^2 - 1\right)^{1/2}} = \pm\, d\lambda \qquad (11.14)$$

$$\text{Where } z = \left(\frac{\Omega}{\Omega - 1}\right)^{1/2} x \qquad (11.15)$$

Set $\quad z = \cosh \alpha \qquad (11.16)$

$$\curvearrowright \quad dz / (z^2 - 1)^{1/2} = d\alpha$$
$$= d \operatorname{arcosh}(z)$$

$$\Rightarrow \frac{1}{\sqrt{\Omega}} \operatorname{arcosh}\left[\left(\frac{\Omega}{\Omega - 1}\right)^{1/2} x\right]$$
$$= \pm (\lambda - \lambda_0) \qquad (11.17)$$

Where λ_0 is that λ-value for which

$$x(\lambda_0) = x_0 = \left(\frac{\Omega - 1}{\Omega}\right)^{1/2} \qquad (11.18)$$

Without loss of generality we may choose $\lambda_0 = 0$ and also choose the positive sign on the right-hand side. Then

$$\chi(\lambda) = \left(\frac{\Omega - 1}{\Omega}\right)^{1/2} \cosh\left(\sqrt{\Omega}\,\lambda\right). \qquad (11.19)$$

Now, from (11.6-7), we have ($k = +1$)

$$\sqrt{\Omega - 1} = c/(H_0 a_0) \qquad (11.20)$$

$$\sqrt{\Omega} = \sqrt{\Lambda/3}\; c/H_0 \qquad (11.21)$$

Hence, with (11.4-5), (11.19) is equivalent to

$$a(t) = \sqrt{\frac{3}{\Lambda}} \cdot \cosh\left(\sqrt{\frac{\Lambda}{3}} \cdot ct\right) \qquad (11.22)$$

describing a "bouncing" universe of positive spatial curvature ("closed") where

$$a(t \to \pm\infty) \longrightarrow \infty \qquad (11.23)$$

and $$\inf_t \{a(t)\} = a(t=0) = \sqrt{\frac{3}{\Lambda}} \qquad (11.24)$$

Moreover

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \sqrt{\frac{\Lambda}{3}} \cdot c \cdot \tanh\left(\sqrt{\frac{\Lambda}{3}}\, ct\right) \qquad (11.25)$$

$$q(t) = -\frac{\ddot{a}(t)\, a(t)}{\dot{a}^2(t)} = -\coth^2\left(\sqrt{\frac{\Lambda}{3}} \cdot ct\right) \qquad (11.26)$$

The spatial manifold $(\hat{M}, \hat{g})$ is of constant sectional curvature $+1$ and hence the slice $t = \text{const}$ of constant sectional curvature $a^{-2}(t)$. If we assume $\hat{M}$ to be complete it must be compact. If, moreover, we assume it to be simply connected – as we always may do, but don't have to – $(\hat{M}, \hat{g})$ must be the unit 3-sphere. Then

$$g = c\,dt \otimes c\,dt - \left(\frac{3}{\Lambda}\right) \cosh^2\left(\sqrt{\frac{\Lambda}{3}}\, ct\right) \times \left\{ d\chi \otimes d\chi + \sin^2(\chi)\left(d\theta \otimes d\theta + \sin^2\theta\, d\varphi \otimes d\varphi\right)\right\} \tag{11.27}$$

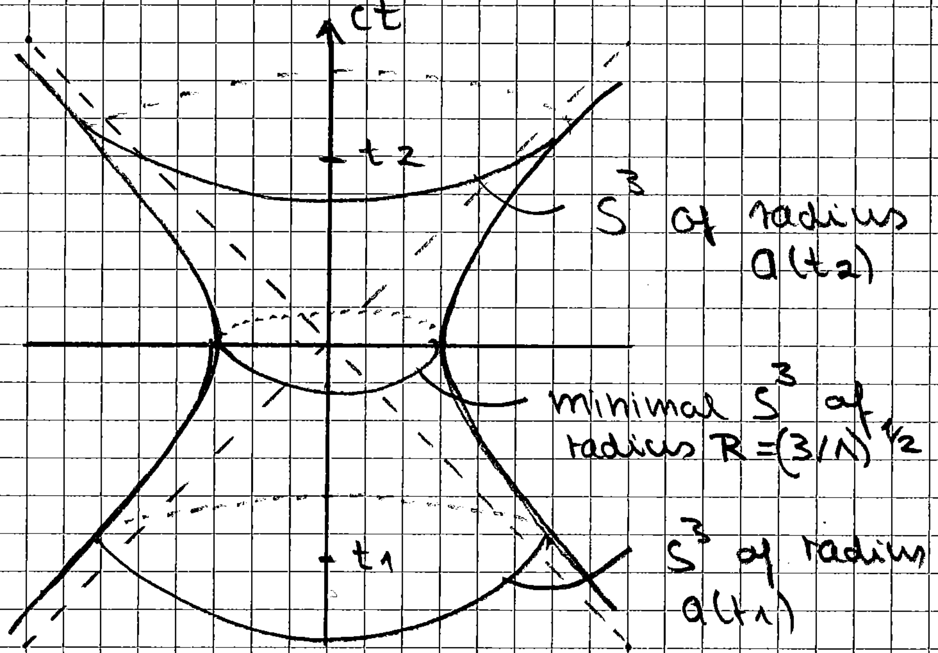
$$= c\,dt \otimes c\,dt - R^2 \cosh^2\left(\frac{ct}{R}\right) \cdot \hat{g}^{(1)}_{S^3} \tag{11.28}$$

where $g^{(1)}_{S^3}$ is the round metric on the unit S^3 of radius 1 and

$$R = (3/\Lambda)^{1/2} \tag{11.29}$$

This is called "de Sitter" spacetime.

We will discuss the geometry of de Sitter later, but it is already clear that it looks like a stack of 3-spheres with some minimal S^3 at the "waist":



(11.30)

$$a(t) = \sqrt{\frac{3}{\Lambda}} \cosh\left(\sqrt{\frac{\Lambda}{3}}\, ct\right)$$

minimal at $t=0$ and reflection symmetric, $a(-t) = a(t)$ and exponential expansion $\sim \exp\left(\sqrt{\frac{\Lambda}{3}}\, ct\right)$ for $t \to \pm\infty$.

Case 2. $\Omega = 1, \; \Omega_x = 0$

$$\left(\frac{dx}{d\lambda}\right) = x^2 \qquad (11.31)$$

$$\rightsquigarrow \quad \frac{dx}{x} = \pm \, d\lambda \qquad (11.32)$$

$$x(\lambda) = x_0 \exp\left(\pm(\lambda - \lambda_0)\right) \qquad (11.33)$$

Let, w.l.o.g. λ_0 such that $\exp(-\lambda_0) = 1/x_0$. Then

$$a(t) = a_0 \exp(H_0 t) \qquad (11.34)$$

From $\Omega_\Lambda = 1$ and (11.6) we infer

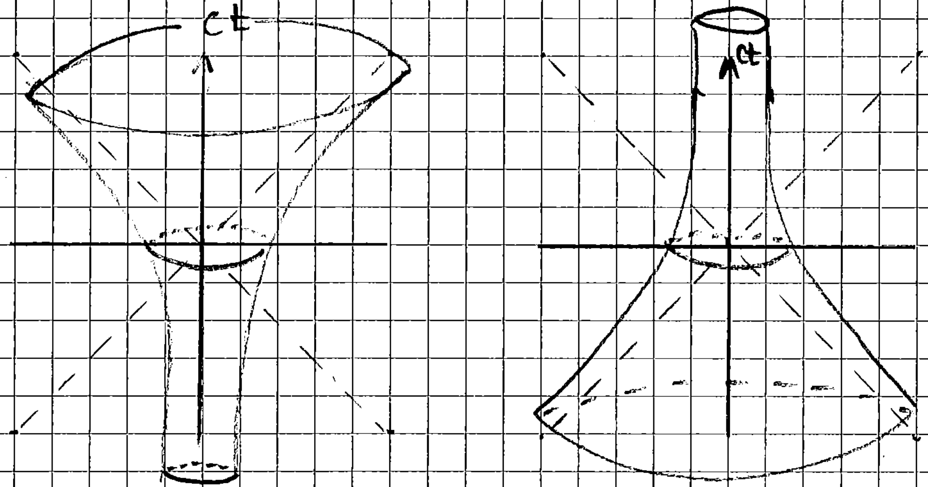
$$H_0 = \sqrt{\frac{\Lambda}{3}} \; c \qquad (11.35)$$

$$\rightsquigarrow \quad a(t) = a_0 \exp\left(\pm \sqrt{\frac{\Lambda}{3}} \cdot ct\right) \qquad (11.36)$$

$$H = \frac{\dot a}{a} = \pm\sqrt{\frac{\Lambda}{3}}\, c = \text{const} \qquad (11.37)$$

$$q = -\frac{\ddot a \, a}{\dot a^2} = -1 = \text{const.} \qquad (11.38)$$

This universe is spatially flat and either in the state of eternal exponentially increasing expansion ("+" - sign) or eternal exponentially decreasing contraction ("−" - sign).



(11.39)

Case 3. $0 < \Omega < 1$, $\Omega k > 0$
i.e. $k = -1$

$$\left(\frac{dx}{d\lambda}\right)^2 - \Omega x^2 = (1-\Omega) \qquad (11.40)$$

$$\frac{dx}{\left(\Omega x^2 + (1-\Omega)\right)^{1/2}} = \pm d\lambda$$

$$\frac{1}{\sqrt{1-\Omega}} \frac{dx}{\left\{\left[\left(\frac{\Omega}{1-\Omega}\right)^{1/2} x\right]^2 + 1\right\}^{1/2}} = \pm d\lambda$$

$$= \frac{1}{\sqrt{\Omega}} \frac{dz}{\left(z^2 + 1\right)^{1/2}} = \pm d\lambda \qquad (11.41)$$

where $z = \left(\frac{\Omega}{1-\Omega}\right)^{1/2} x$. (11.42)

Set $z = \sinh \alpha$ (11.43)

$\rightsquigarrow$ $dz / (z^2+1)^{1/2} = d\alpha$
$= d \operatorname{arsinh}(z)$

$$\Rightarrow \frac{1}{\sqrt{\Omega}} \operatorname{arsinh}\left[\left(\frac{\Omega}{1-\Omega}\right)^{1/2} x\right] = \pm(\lambda - \lambda_0) \qquad (11.44)$$

where λ_0 is the λ-value for which $x(\lambda_0) = 0$ ("Big-Bang"). (11.45)

Without loss of generality we may choose $\lambda_0 = 0$ and also the positive sign on the right-hand side; then $\lambda \geq 0$, for $\chi \geq 0$. Hence we have

$$\chi(\lambda) = \left(\frac{1-\Omega}{\Omega}\right)^{1/2} \sinh\left(\sqrt{\Omega}\,\lambda\right) \qquad (11.46)$$

Again, from (11.6-7) we have ($k=-1$)

$$\sqrt{1-\Omega} = c/(a_0 H_0) \qquad (11.47)$$

$$\sqrt{\Omega} = \sqrt{\Lambda/3}\; c/H_0 \qquad (11.48)$$

Hence, with (11.4-5), (11.46) is equivalent to

$$a(t) = \sqrt{\frac{3}{\Lambda}} \cdot \sinh\left(\sqrt{\frac{\Lambda}{3}} \cdot ct\right) \qquad (11.49)$$

describing a "Big-Bang" universe of negative spatial curvature ("open") where

$$a(t\to\infty) = \infty, \quad a(t \searrow 0) = 0. \qquad (11.50)$$

Moreover

$$H(t) = \frac{\dot a(t)}{a(t)} = \sqrt{\frac{\Lambda}{3}} \cdot c \cdot \coth\left(\sqrt{\frac{\Lambda}{3}} \cdot ct\right) \qquad (11.51)$$

$$q(t) = -\frac{\ddot a(t)\, a(t)}{\dot a^2(t)} = -\tanh^2\left(\sqrt{\frac{\Lambda}{3}} \cdot ct\right) \qquad (11.52)$$

The spatial manifold $(\hat{M}, \hat{g})$ is of constant sectional curvature -1 and hence the slice $t = \text{const.}$ of constant sectional curvature $-a^{-2}(t)$. If we assume $\hat{M}$ to be complete we now cannot conclude that $\hat{M}$ must be compact, unlike the $k = +1$ case, but it <u>may be</u> compact.

Compare this to 2-dimensional case: A 2-dim manifold of constant pos. curvature (Sectional = Gauß) must be compact if it is complete. Hence it must be S^2. It cannot be any other Riemann surface because of the <u>Gauß-Bonnet-Theorem</u>:

$$\int_\Sigma K \, dA = 2\pi \, \chi(\Sigma)$$

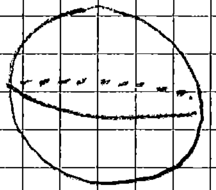
$$= 4\pi(1-g) \qquad (11.53)$$

$\chi(\Sigma)$ = Euler characteristic of Σ (top. invariant)

$$= 2(1-g) \qquad (11.54)$$

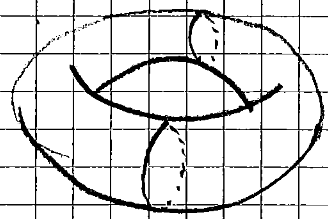
g = genus of Σ.

S^2:



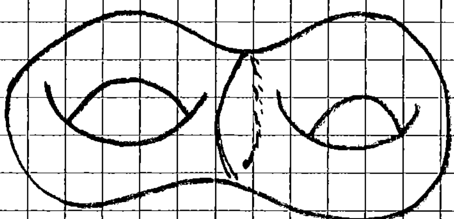
$$\chi = 2, \quad g = 0 \tag{11.55}$$

T^2:



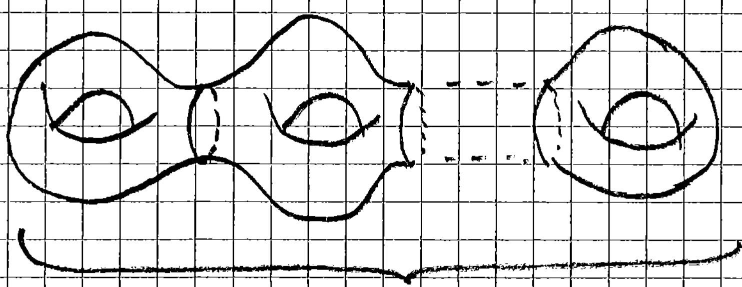
$$\chi = 0, \quad g = 1 \tag{11.56}$$

$T^2 \# T^2$:



$$\chi = -2, \quad g = 2 \tag{11.57}$$

$\overset{n}{\underset{i=1}{\#}} T^2$:



connected sum of n tori T^2

$$\chi = 2(1-n), \quad g = n \tag{11.58}$$

GBT $\Rightarrow$ For $g \neq 0$ Σ cannot have metric of const. pos. curvature.

Note that means that the Gauß-Bonnet-Theorem provides topological obstructions against metrics of everywhere positive curvature: only the 2-sphere can have those. It also provides a - much weaker! - topological obstruction against metrics of everywhere negative curvature: all closed manifolds can have them except S^2 and T^2. Hence almost all can have $K < 0$, but almost none can have $K > 0$. Only one can have $K \equiv 0$.

In 3-dimensions a similar result holds for the scalar curvature: "Most" admit $R < 0$ metrics, "few" admit $R > 0$ metrics. And there are countably infinite pairwise non-homeomorphic closed 3-dim. manifolds admitting metrics with constant negative sectional curvature - so called "hyperbolic manifolds". Hence it is far from true that "open" cosmologies must have non-compact spaces ($t =$ const. slices).

Case 4. $\Omega = 0$, $\Omega_K = 1$, $k = -1$

$$\left(\frac{dX}{d\lambda}\right)^2 = 1 \iff X(\lambda) = \lambda \tag{11.59}$$

(choosing $\lambda_0 = 0$). Hence

$$a(t) = a_0 H_0 t \tag{11.60}$$

$$\rightsquigarrow \quad H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{1}{t} \tag{11.61}$$

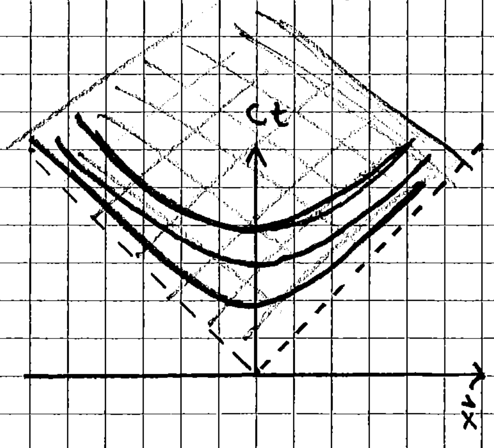
$$H_0 = 1/t_0$$

$$\Rightarrow \quad a(t) = a_0 \frac{t}{t_0} = ct \tag{11.62}$$

from $1 = \Omega_K = \frac{c^2}{H_0^2 a_0^2} \rightsquigarrow \frac{a_0}{t_0} = c$. The metric is:

$$g = c\,dt \otimes c\,dt - a^2(t)\,\hat{g} \tag{11.63}$$

$$\hat{g} = dX \otimes dX + \sinh^2(X)\,\{ d\theta \otimes d\theta + \sin^2\theta\, d\varphi \otimes d\varphi \} \tag{11.64}$$



This is the Milne-Universe that we already encountered on Sheet 2, Problem 3. Here spacetime is the "upper Wedge" in Minkowski-space.

Case 5: $\Omega < 0$, $\Omega_k > 1$, $k = -1$

$$\frac{dx}{d\lambda} = \Omega x^2 + \underbrace{(1-\Omega)}_{>1} \tag{11.65}$$

$$\frac{dx}{\left[(1-\Omega) + \Omega x^2\right]^{1/2}} = \pm\, d\lambda$$

$$\frac{1}{(1-\Omega)^{1/2}} \int \frac{dx}{\left[1 - \frac{|\Omega|}{1-\Omega} x^2\right]^{1/2}} = \pm\, d\lambda$$

$$= \frac{1}{\sqrt{|\Omega|}} \int \frac{dz}{(1-z^2)^{1/2}} = \pm\, d\lambda \tag{11.66}$$

where $z := \left(\frac{|\Omega|}{1-\Omega}\right)^{1/2} x$ (11.67)

Set $z = \sin\alpha$

$\rightsquigarrow$ $dz / (1-z^2)^{1/2} = d\alpha$
$= d \arcsin(z)$

$$\Rightarrow \frac{1}{\sqrt{|\Omega|}} \arcsin\left(\frac{|\Omega|}{1-\Omega} x\right) = \pm(\lambda - \lambda_0) \tag{11.68}$$

where $x(\lambda_0) = 0$.

Choose w.l.o.g. positive sign on r.h.s and $\lambda_0 = 0$.

$$\Rightarrow \quad x(\lambda) = \left(\frac{1-\Omega}{|\Omega|}\right)^{1/2} \sin\left(\sqrt{|\Omega|}\,\lambda\right) \qquad (11.69)$$

Using (11.6-7),

$$\sqrt{|\Omega|} = \sqrt{\frac{|\Lambda|}{3}}\,\frac{c}{H_0} \qquad (11.70a)$$

$$\sqrt{1-\Omega} = \frac{c}{H_0 a_0} \qquad (11.70b)$$

and (11.4), we get (recall $x = \frac{a}{a_0}$),

$$a(t) = \sqrt{\frac{3}{|\Lambda|}} \cdot \sin\left(\sqrt{\frac{|\Lambda|}{3}}\,ct\right) \qquad (11.71)$$

again describing a "Big-Bang" universe with negative spatial curvature ("open") which, in contrast to case 3, i.e. (11.49), is recollapsing. The maximal scale factor

$$a_{max} = \sqrt{\frac{3}{|\Lambda|}} \qquad (11.72)$$

is reached at

$$t_{max} = \sqrt{\frac{3}{|\Lambda|}}\,\frac{\pi}{2c} = \frac{\pi \cdot a_{max}}{2c} \qquad (11.73)$$

which is half the "life time"

$$t_* = \pi\,\frac{a_{max}}{c}. \qquad (11.74)$$

The metric is

$$g = c\,dt \otimes c\,dt$$

$$- \left(\frac{3}{\Lambda}\right) \sin^2\left(\sqrt{\frac{\Lambda}{3}}\, ct\right) \Big\{ d\chi \otimes d\chi$$

$$+ \sinh^2(\chi)\left(d\theta \otimes d\theta + \sin^2\theta\, d\varphi \otimes d\varphi\right)\Big\} \quad (11.75)$$

Summary of $a(t)$ for all 5 cases

$$a(t) = \begin{cases} \left(\frac{3}{\Lambda}\right)^{1/2} \cosh\left[\left(\frac{\Lambda}{3}\right)^{1/2} ct\right] & \Omega_\Lambda > 1,\ k = 1 \\ a_0 \exp\left[\pm\left(\frac{\Lambda}{3}\right)^{1/2} ct\right] & \Omega_\Lambda = 1,\ k = 0 \\ \left(\frac{3}{\Lambda}\right)^{1/2} \sinh\left[\left(\frac{\Lambda}{3}\right)^{1/2} ct\right] & 0 < \Omega_\Lambda < 1,\ k = -1 \\ ct & \Omega_\Lambda = 0,\ k = -1 \\ \left(\frac{3}{|\Lambda|}\right)^{1/2} \sin\left[\left(\frac{|\Lambda|}{3}\right)^{1/2} ct\right] & \Omega_\Lambda < 0,\ k = -1 \end{cases} \quad (11.76)$$

Note that for $k = +1$ (case 1) the spatial coordinate χ is restricted to $[0, \pi]$ (polar-angle of 3-sphere), whereas in other cases there is no a priori restriction to χ.

We are interested in the possible existence of horizons (particle and event). For this we have to calculate the integrals

$$L(t_0, t_1) := a_0 \int_{t_1}^{t_0} \frac{c\,dt}{a(t)}$$

or just

$$h(t_0, t_1) = \int_{t_1}^{t_0} \frac{c\,dt}{a(t)} \qquad (11.77)$$

In a "Big-Bang" cosmology where $a(t) = 0$ for $t = t_*$ a particle horizon exists at time t_0 if

$$\lim_{t_1 \to t_*} h(t_0, t_1) < \Delta\chi_{max} \qquad (11.78)$$

where $\Delta\chi_{max}$ is the maximal range of χ, either π in the $k = +1$ case or else ∞.

An event horizon exists if a particle horizon exists for all t_0, including $t_0 \to t_{max}$, where t_{max} = finite for recollapsing universe ($t_{max} = t_{BC}$), or else ∞.

Case 1

$$h(t_0, t_1) = \int_{t_1}^{t_0} \frac{c\,dt}{\left(\frac{3}{\Lambda}\right)^{1/2} \cosh\left[\left(\frac{\Lambda}{3}\right)^{1/2} ct\right]}$$

$$= \int_{z_1}^{z_0} \frac{dz}{\cosh(z)} \qquad (11.79)$$

where $z = \left(\frac{\Lambda}{3}\right)^{1/2} ct$

$$h(t_0, t_1) = 2 \arctan\left(\tanh\left(\frac{z}{2}\right)\right)\Big|_{z_1}^{z_0} \qquad (11.80)$$

As this is not a Big-Bang cosmology and $k = +1$ we have $\chi_{max} = \pi$. For the limit of z_1 we take $-\infty$ so that, e.g.,

$$h(0, -\infty) = 2 \arctan(0) - 2 \arctan(-1) = \pi/2 \qquad (11.81)$$

covering half the 3-sphere. Hence

$$h(\infty, -\infty) = \pi \ \hat{=} \text{ whole } S^3. \qquad (11.82)$$

This means that an observer at t_0 sees only half the particles (galaxies), but waiting long enough for $t_0 \to \infty$ eventually all particles will be seen. This does not mean that there is no event horizon, as there may be events not in or on the backward lightcone of an observer at $t_0 \to \infty$.

Case 2.

$$h(t_0, t_1) = \int_{t_1}^{t_0} \frac{c\,dt}{a_0 \exp\left[\pm \left(\frac{\Lambda}{3}\right)^{1/2} ct\right]}$$

$$= \frac{(3/\Lambda)^{1/2}}{a_0} \int_{z_1}^{z_0} \exp(\mp z)\, dz$$

$$= \mp \frac{(3/\Lambda)^{1/2}}{a_0} \left(e^{\mp z_0} - e^{\mp z_1}\right) \qquad (11.83)$$

So for the exponentially expanding case (upper sign

$$h(t_0, t_1) = \frac{(3/\Lambda)^{1/2}}{a_0} \left(e^{-z_1} - e^{-z_0}\right) \qquad (11.84)$$

$\to \infty$ for $z_1 \to -\infty$; hence no particle horizon at any time.

In the exponentially contracting case (lower sign)

$$h(t_0,t_1) = \frac{(3/\Lambda)^{1/2}}{a_0}\left(e^{z_0}-e^{z_1}\right) \qquad (11.85)$$

which is finite for $z_1 \to -\infty$ (particle horizon at any finite time) and diverges for $z_0 \to \infty$, i.e. all particles will eventually be seen.

Case 3.

$$h(t_0,t_1) = \int_{t_1}^{t_0} \frac{c\,dt}{\left(\frac{3}{\Lambda}\right)^{1/2} \sinh\left[\left(\frac{\Lambda}{3}\right)^{1/2} ct\right]}$$

$$= \int_{z_1}^{z_0} \frac{dz}{\sinh(z)}$$

$$= \ln\left(\tanh\left(\frac{z}{2}\right)\right)\Big|_{z_1}^{z_0}$$

$$= \ln\left\{\frac{\tanh(z_0/2)}{\tanh(z_1/2)}\right\} \qquad (11.86)$$

This is a Big-Bang cosmology with $t_* = 0$. Since $\tan(z_*/2) = 0$ and $-\ln(\tan(z_1 \to z_*/2)) \to +\infty$

there is no particle horizon here.

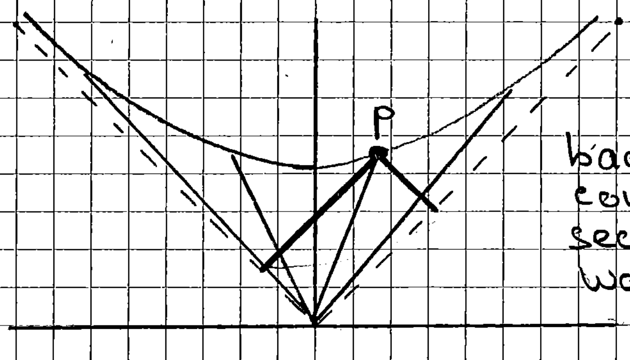
Case 4

$$h(t_1, t_0) = \int_{t_0}^{t_1} \frac{c\,dt}{c\,t} = \ln\left(\frac{t_1}{t_0}\right) \qquad (11.87)$$

Again this is a Big-Bang cosmology with $t_* = 0$ and

$$\lim_{t_1 \to t_*} h(t_0, t_1) \to \infty \qquad (11.88)$$

hence no particle horizon. This is in fact obvious from the geometric embedding of Milne into Minkowski



(11.89)

backward light-cone at p intersects any other worldline in Milne universe.

Case 5

$$h(t_1,t_0)=\int_{t_1}^{t_0}\frac{c\,dt}{\left(\frac{3}{|\Lambda|}\right)^{1/2}\sin\left[\left(\frac{|\Lambda|}{3}\right)^{1/2}ct\right]}$$

$$=\int_{z_1}^{z_0}\frac{dz}{\sin(z)}$$

$$=-\ln\left(\frac{1+\cos(z)}{\sin(z)}\right)\Bigg|_{z_1}^{z_0} \qquad (11.90)$$

This is a (Big-Bang) - (Big-Crunch) cosmology with BB at $z_1=0$.
Hence

$$\lim_{t_1\to t_*}\left(h(t_1,t_0)\right)=\infty \qquad (11.91)$$

$\Rightarrow$ no particle horizon