Lecture 12: (anti)-de Sitter manifolds

As we have already seen on Sheet 2 Problem 3, we can identify the Milne univers (M,g) with the "upper wedge" of Minkowski-space. This means that there is an isometric embedding of (M,g) into (M^4,η) = Minkowski space

$$(M,g) \hookrightarrow (M^4,\eta) \tag{12.1}$$

so that (M,g) is identified with the subset

$$M := \{(X^0,\vec{X}) \in M^4 : X^0 > \|\vec{X}\|\} \tag{12.2}$$

and $g = \eta\big|_M$ (induced metric). (12.3)

It is natural to ask whether and how cases 1-3 and case 5 can also be identified with suitable subsets of geometrically simple manifolds. This is indeed the case and the somewhat surprising fact is that cases 1.-3. are all isometrically embeddable in <u>the same</u> manifold, called <u>de Sitter space</u>. Case 5 is related but different, being a subset of

anti - de Sitter space

De Sitter Space

The following works in all dimensions $n \geq 2$.

Let (M^{n+1}, η) be $(n+1)$-dim. Minkowski space with standard coordinates

$$(X^0, X^1, \ldots, X^n) = (X^0, \vec{X}), \quad \vec{X} \in \mathbb{R}^n \qquad (12.4)$$

and metric

$$\eta = dX^0 \otimes dX^0 - \sum_{a=1}^{n} dX^a \otimes dX^a$$

$$= dX^0 \otimes dX^0 - d\vec{X} \dot{\otimes} d\vec{X} \qquad (12.5)$$

In M^{n+1} we consider the n-dimensional one-sheeted (i.e. connected) hyperboloid of all vectors $(M^{n+1} \cong \mathbb{R}^{n+1})$ of Minkowski length-square $-R^2$:

$$H^n_R := \{ (X^0, \vec{X}) \in \mathbb{R}^{n+1} : (X^0)^2 - \|\vec{X}\|^2 = -R^2 \} \qquad (12.6)$$

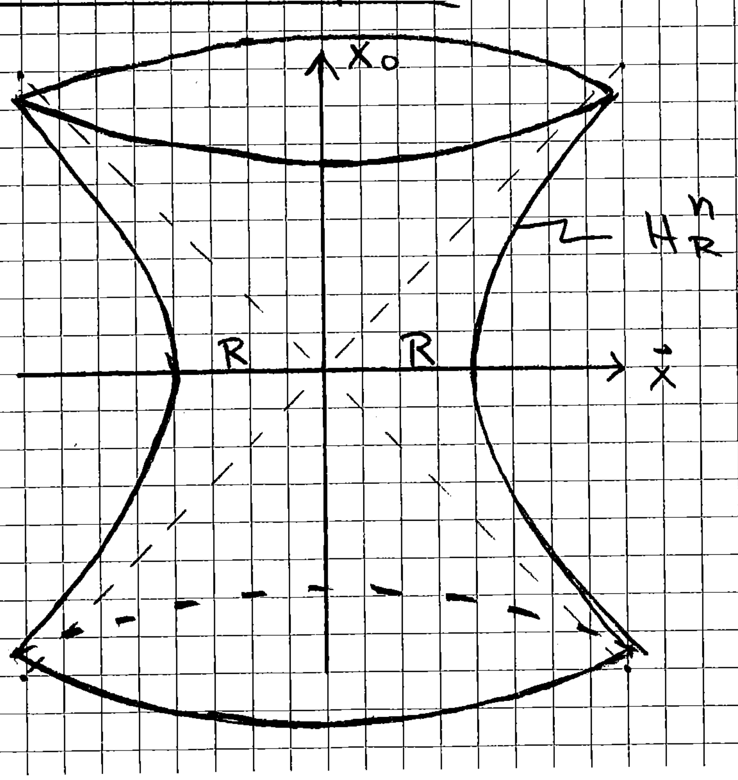
Given $X = (X^0, \vec{X}) \in H^n_R \subset \mathbb{R}^{n+1}$, the n-dimensional plane orthogonal to X can be identified with the tangent space $T_X H^n_R$:

$$T_X H^n_R = X^\perp := \{ y \in \mathbb{R}^{n+1} : \eta(X,y) = 0 \} \qquad (12.7)$$

As X is spacelike in $(\mathbb{R}^{n+1}, \eta)$ its orthogonal complement is timelike. Hence H^n_R is a Lorentz-manifold of dimension n. Together with its induced metric, i.e. for $V, W \in T_X H^n_R \cong X^\perp$

$$g(V,W) = \eta|_{X^\perp}(V,W), \qquad (12.8)$$

this n-dimensional Lorentz-manifold is called <u>the n-dimensional de Sitter space.</u>



(12.9)

The metric of de Sitter space follows from

$$g = \eta \big|_{H^n_R}$$

$$= \eta_{\alpha\beta}\, dX^\alpha \otimes dX^\beta \big|_{H^n_R} \qquad (12.10)$$

where on H^n_R we have

$$\eta_{\alpha\beta} X^\alpha X^\beta = -R^2 \qquad (12.11)$$

There are various ways to choose coordinates on this manifold, partly with restricted domains of definition. We will show that there exist coordinates that put the metric into the form of case 1, 2, or 3. In addition, there is a fourth coordinate system that gives the metric a static form with cosmological horizon. This is the form de Sitter originally presented the metric in 1917.

Case 1: The closed form

$$X^0 = R \sinh(ct/R) \qquad (12.12a)$$

$$X^a = R \cosh(ct/R)\, Z^a, \quad 1 \leq a \leq n \qquad (12.12b)$$

where Z^a are coordinates on the unit $(n-1)$-sphere, i.e.

$$\sum_{a=1}^{n} Z^a Z^a = 1 \qquad (12.12c)$$

Hence the map

$$i : \mathbb{R} \times S^{n-1} \hookrightarrow H^n_R$$

$$(t, Z^a) \mapsto (X^0(t), X^a(t, Z^a)) \qquad (12.13)$$

is an isometric embedding that is actually also surjective, hence an isometry. The coordinate chart defined by (12.12) is a global one (if one regards the Z^a as global coordinates on S^{n-1}, which is not quite true). From (12.12a, b) we get

$$dX^0 = \cosh(ct/R)\, c\, dt \qquad (12.14a)$$

$$dX^a = \sinh(ct/R)\, Z^a\, c\, dt + R \cosh(ct/R)\, dZ^a \qquad (12.14b)$$

Writing

$$\left.\begin{aligned}\vec{X} &:= (X^1, \cdots, X^n) \\ \vec{Z} &:= (Z^1, \cdots, Z^n)\end{aligned}\right\} \quad (12.15)$$

We have

$$d\vec{X} = \sinh(ct/R)\,\vec{Z}\;c\,dt + R\cosh(ct/R)\;d\vec{Z} \qquad (12.16)$$

and

$$d\vec{X} \dot{\otimes} d\vec{X} := \sum_{a=1}^{n} dX^a \otimes dX^a = \sinh^2(ct/R)\;c\,dt \otimes c\,dt + \cosh^2(ct/R)\,R^2\;d\vec{Z} \dot{\otimes} d\vec{Z} \qquad (12.17)$$

Note that

$$\hat{g}^{(1)}_{S^{n-1}} := d\vec{Z} \dot{\otimes} d\vec{Z} \qquad (12.18)$$

is the metric on the unit $(n-1)$ sphere S_1^{n-1}.

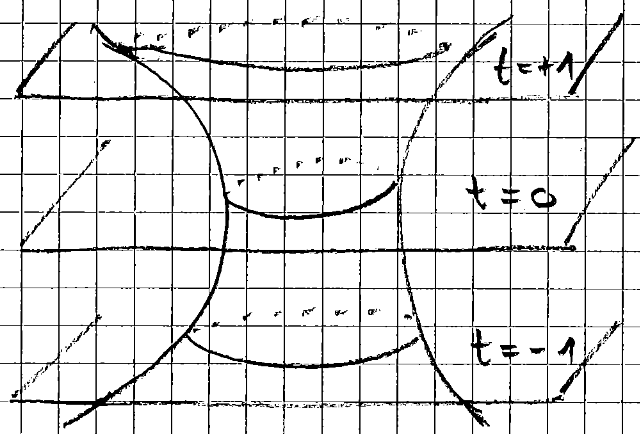
Hence (11.101) gives

$$g = dX^0 \otimes dX^0 - d\vec{X} \dot{\otimes} d\vec{X} = c\,dt \otimes c\,dt - R^2\cosh^2(ct/R)\,\hat{g}^{(1)}_{S^{n-1}} \qquad (12.19)$$

If we compare this with (11.28) we see that for $n=4$ we just get case 1 for

$$R = \left(\frac{3}{\Lambda}\right)^{1/2} \tag{12.20}$$

Note that in this coordinates $(t, \vec{z})$ the hypersurfaces of constant time t are – according to (12.12a) – just the hyperplanes of constant X^0 in $\mathbb{R}^{n+1}$



Now we recall the curvature components of general FLRW-metrics in the orthonormal frame

$$\left.\begin{aligned} \theta^0 &= c\, dt \\ \theta^a &= a(t)\, \hat{\theta}^a \end{aligned}\right\} \tag{12.21}$$

in which the metric is

$$g = \theta^0 \otimes \theta^0 - a^2(t) \underbrace{\sum_{a=1}^{3} \hat{\theta}^a \otimes \hat{\theta}^a}_{\hat{g}} \tag{12.22}$$

where $\hat{g}$ has constant sectional curvature. Then (4.78) reads

$$R_{0a0b} = \frac{\ddot{a}}{c^2 a}\,\delta_{ab} \qquad (12.23a)$$

$$R_{abab} = -\left(k + \frac{\dot{a}^2}{c^2}\right)/\,a^2 \quad (a \neq b) \qquad (12.23b)$$

In our case

$$a(t) = R\cosh(ct/R) \qquad (12.24)$$

so that

$$\left.\begin{aligned} \dot{a}(t) &= c\,\sinh(ct/R) \\ \ddot{a}(t) &= (c^2/R)\cosh(ct/R) \end{aligned}\right\} \qquad (12.25)$$

$$\Rightarrow \ddot{a}/(ac^2) = 1/R^2 \qquad (12.26)$$

and since $k = +1$ $(\hat{\Sigma} = S^3_1)$

$$\begin{aligned} (k + \dot{a}^2/c^2) &= 1 + \sinh^2(ct/R) \\ &= \cosh^2(ct/R) \\ &= \frac{1}{R^2}\,a^2 \end{aligned} \qquad (12.27)$$

$$\begin{aligned} R_{0a0b} &= R^{-2}\,\delta_{ab} \\ &= -\frac{1}{R^2}\left(g_{00}\,g_{ab} - g_{0b}\,g_{0a}\right) \end{aligned} \qquad (12.28)$$

The second line holds true because in orthonormal frame have

$$\left.\begin{aligned} g_{00} &= 1 , \\ g_{ab} &= -\delta_{ab}, \\ g_{0a} &= g_{0b} = 0. \end{aligned}\right\} \quad (12.29)$$

Moreover, for $a \neq b$

$$\begin{aligned} R_{abab} &= -\frac{1}{R^2} \\ &= -\frac{1}{R^2}\left(g_{aa}\, g_{bb} - g_{ab}\, g_{ab}\right) \quad (12.30) \end{aligned}$$

Hence we have shown that for de Sitter space

$$R_{\alpha\beta\mu\nu} = -\frac{1}{R^2}\left(g_{\alpha\mu}\, g_{\beta\nu} - g_{\alpha\nu}\, g_{\beta\mu}\right) \quad (12.31)$$

i.e. de Sitter space is itself, as spacetime-manifold, of constant sectional curvature $-1/R^2$. The minus-sign is due to our "mostly minus" sign convention (Note that since $R^{\alpha}{}_{\beta\mu\nu}$ does not change sign under $g \mapsto -g$, the covariant components $R_{\alpha\beta\mu\nu}$ do.)

From (11.122) we get, taking traces, and generalising to arbitrary space-time dimensions n,

$$R_{\alpha\beta} = -\frac{1}{R^2}\left(n\, g_{\alpha\beta} - g_{\alpha\beta} \right)$$

$$= -\frac{(n-1)}{R^2}\, g_{\alpha\beta} \qquad (12.32)$$

$$R = -\frac{n(n-1)}{R^2} \qquad (12.33)$$

Ricci scalar — radius of de Sitter

$$\leadsto \quad G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta}$$

$$= -\frac{n-1}{R^2}\, g_{\alpha\beta} \left(1 - \frac{n}{2} \right)$$

$$= \frac{(n-1)(n-2)}{2R^2}\, g_{\alpha\beta} \qquad (12.34)$$

Hence

$$G_{\alpha\beta} - g_{\alpha\beta}\Lambda = 0 \qquad (12.35)$$

if

$$\Lambda = \frac{(n-1)(n-2)}{2R^2} \qquad (12.36)$$

or

$$R = \left(\frac{(n-1)(n-2)}{2\Lambda} \right)^{1/2} \qquad (12.37)$$

For $n=4$ this is just (11.111).

Case 2: The flat form

$$X^0 = R \sinh(ct/R) + \frac{r^2}{2R} \exp(ct/R) \qquad (12.38a)$$

$$X^1 = R \cosh(ct/R) - \frac{r^2}{2R} \exp(ct/R) \qquad (12.38b)$$

$$X^A = \exp(ct/R)\, y^A \,, \quad 2 \leq A \leq n \qquad (12.38c)$$

where

$$r^2 := \sum_{A=2}^{n} (y^A)^2 \qquad (12.38d)$$

and the range of y^A is $(-\infty, \infty)$ each.

This defines a map

$$i: \mathbb{R} \times \mathbb{R}^{n-1} \hookrightarrow H^n_R$$

$$(t, y^A) \mapsto (X^0(t,r), X^1(t,r), X^A(t, y^A)) \qquad (12.39)$$

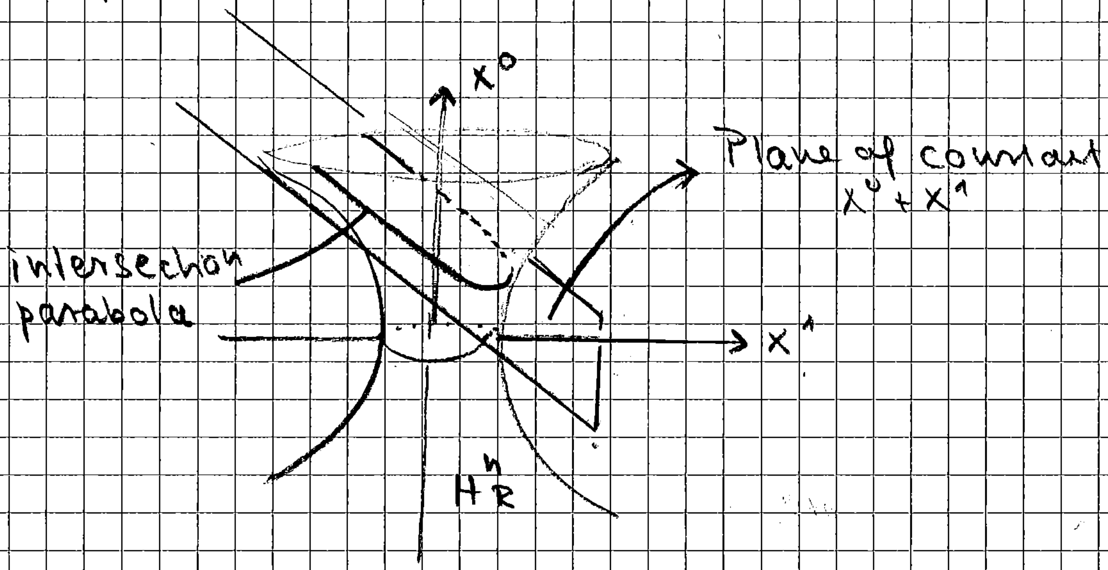
which is an isometric embedding that is <u>not</u> surjective. This is already seen from (12.38 a,b):

$$X^0 + X^1 = R\,[\sinh(ct/R) + \cosh(ct/R)]$$

$$= R \exp(ct/R) > 0 \qquad (12.40)$$

Hence the image lies entirely on that part of H^n_R which is contained in the half-space $X^0 + X^1 > 0$ in $\mathbb{R}^{n+1}$.

The hypersurfaces of constant t correspond to the intersections of the hyperplanes of constant $X^0 + X^1$ with H^n_R. The former are lightlike, whose intersections with the timelike tangent spaces $T_X H^n_R = \{X\}^\perp$ are spacelike.



(12.41)

Now, writing

$$f(t, r) := \frac{r^2}{2R} \exp(ct/R) \qquad (12.42)$$

we have from (11.129 a, b):

$$dX^0 = \cosh(ct/R)\, c\,dt + df \qquad (12.43a)$$
$$dX^1 = \sinh(ct/R)\, c\,dt - df \qquad (12.43b)$$

$$dX^0 \otimes dX^0 - dX^1 \otimes dX^1 = c\,dt \otimes c\,dt$$
$$+ \left(\cosh(ct/R) + \sinh(ct/R)\right)$$
$$\left(c\,dt \otimes df + df \otimes c\,dt\right)$$
$$= c\,dt \otimes c\,dt + \exp(ct/R)\left(c\,dt \otimes df + df \otimes c\,dt\right) \qquad (12.44)$$

Now,

$$df = \frac{r^2}{2R^2} \exp(ct/R)\, c\,dt + \frac{1}{2R} \exp(ct/R)\, d(r^2) \tag{12.45}$$

hence

$$\exp(ct/R)\,(c\,dt \otimes df + df \otimes c\,dt)$$
$$= \exp(2ct/R)\Big[\frac{r^2}{R^2}\, c\,dt \otimes c\,dt + \frac{1}{2R}\big(c\,dt \otimes d(r^2) + d(r^2) \otimes c\,dt\big)\Big] \tag{12.46}$$

On the other hand

$$dX^A = \exp(ct/R)\left[dy^A + (y^A/R)\, c\,dt\right]$$

$$\Rightarrow \sum_{A=2}^{n} dX^A \otimes dX^A = \exp(2ct/R)\Big\{ \sum_{A=2}^{n} dy^A \otimes dy^A + \frac{r^2}{R^2}\, c\,dt \otimes c\,dt + \frac{1}{2R}\big(c\,dt \otimes d(r^2) + d(r^2) \otimes c\,dt\big)\Big\} \tag{12.47}$$

where we used

$$\sum_{A=2}^{n} y^A\, dy^A = \frac{1}{2}\, d(r^2) \tag{12.48}$$

The 2nd and 3rd term in (12.47) will cancel the 1st and 2nd term of (12.46).

Hence, using (12.44, 46, 47) we get

$$g = dx^0 \otimes dx^0 - dx^1 \otimes dx^1 - \sum_{A=2}^{n} dx^A \otimes dx^A$$

$$= c\,dt \otimes c\,dt - \exp(2ct/R)\,\hat{g} \qquad (12.49a)$$

where $\hat{g} = \sum_{A=2}^{n} dy^A \otimes dy^A$ (12.49b)

is the $(n-1)$-dimensional flat metric on $\mathbb{R}^{n-1}$.

Hence up to a constant scale a_0, which can always be absorbed into the coordinates y^A, (12.49a) just corresponds to case 2 in (11.76) with

$$a(t) = a_0 \exp(ct/R)$$

$$= a_0 \exp\left((\Lambda/3)^{1/2}\, ct\right) \qquad (12.50)$$

so that again

$$R = \left(\frac{3}{\Lambda}\right)^{1/2} \qquad (12.51)$$

The negative sign in the exponential is obtained by $t \mapsto -t$.

Case 3: The open form

$$X^0 = R \sinh(ct/R) \cosh(\xi) \qquad (12.52a)$$

$$X^1 = R \cosh(ct/R) \qquad (12.52b)$$

$$X^A = R \sinh(ct/R) \sinh(\xi)\, Z^A \qquad (12.52c)$$

$$\text{for} \quad 2 \leq A \leq n$$

and the Z^A are coordinates on the unit $(n-2)$-sphere, satisfying

$$\sum_{A=2}^{n} Z^A Z^A = 1. \qquad (12.52d)$$

Here (ξ, Z^A) coordinatise a manifold $\hat{\Sigma} \cong \mathbb{R}^{n-1}$ of constant negative sectional curvature, so that we have a map

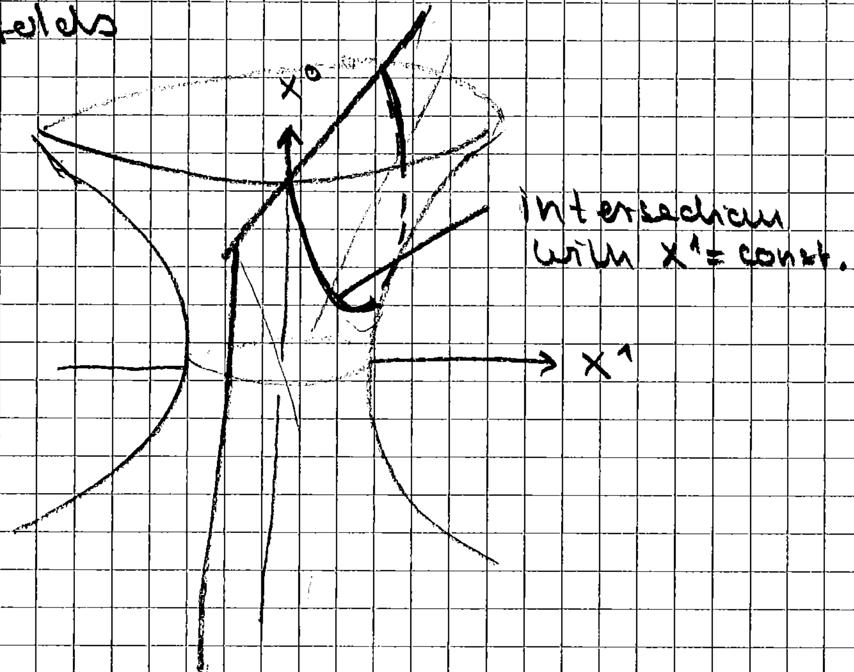
$$i : \mathbb{R} \times \mathbb{R}^{n-1} \hookrightarrow H^n_R$$

$$(t, \xi, Z^A) \mapsto \left(X^0(t,\xi), X^1(t), X^A(t,\xi,Z^A)\right) \qquad (12.52e)$$

Again this is an isometric embedding that is <u>not</u> surjective, as seen, e.g., from $X^1 > 0$ (12.52b).

The hypersurfaces of constant time t are the intersections of H^n_R with the hyperplanes of constant X^1, which are timelike but intersect H^n_R

in spacelike hyperboloidal submanifolds



(12.53)

This is easy to see since if in the defining equation for H^n_R,

$$(X^0)^2 - \sum_{a=1}^{n} (X^a)^2 = -R^2$$

$$(X^0)^2 - (X^1)^2 - \sum_{A=2}^{n} (X^A)^2 = -R^2 \qquad (12.54)$$

we set X^1 = const $> R$, we get

$$(X^0)^2 - \sum_{A=2}^{n} (X^A)^2 = -R^2 + (X^1)^2 > 0 \qquad (12.55)$$

which is an equation for a two-sheeted hyperboloid ("mass shell") with the two sheets corresponding to positive and negative X^0-values respectively. Note $X^0 > 0$ for $t > 0$.

Now,

$$dX^0 = \cosh(ct/R)\cosh(\xi)\, c\,dt$$

$$+ R \sinh(ct/R)\sinh(\xi)\, d\xi \qquad (12.56)$$

$$dX^1 = \sinh(ct/R)\, c\,dt \qquad (12.57)$$

$$dX^A = \cosh(ct/R)\sinh(\xi) Z^A\, c\,dt$$

$$+ R \sinh(ct/R)\cosh(\xi) Z^A\, d\xi$$

$$+ R \sinh(ct/R)\sinh(\xi)\, dZ^A \qquad (12.58)$$

We use again $\sum Z^A\, dZ^A = 0$ and get for

$$\sum_{A=2}^{n} dX^A \otimes dX^A$$

$$= \cosh^2(ct/R)\sinh^2(\xi)\, c\,dt \otimes c\,dt$$

$$+ R^2 \sinh^2(ct/R)\cosh^2(\xi)\, d\xi \otimes d\xi$$

$$+ R\left(\cosh(ct/R)\sinh(ct/R)\sinh(\xi)\cosh(\xi)\right)$$

$$\times \left(c\,dt \otimes d\xi + d\xi \otimes c\,dt\right)$$

$$+ R^2 \sinh^2(ct/R)\sinh^2(\xi) \sum dZ^A \otimes dZ^A \qquad (12.59)$$

The cross-term $\sim (c\,dt \otimes d\xi + d\xi \otimes c\,dt)$ is cancelled by the corresponding cross-term from $dx^0 \otimes dx^0$. Then

$$dx^0 \otimes dx^0 - dx^1 \otimes dx^1 - \sum_{A=2}^{n} dx^A \otimes dx^A$$

$$= c\,dt \otimes c\,dt - R^2 \sinh^2(ct/R)\, d\xi \otimes d\xi$$

$$= R^2 \sinh^2(ct/R) \sinh^2(\xi) \sum dz^A \otimes dz^A. \qquad (12.60)$$

With $\sum dz^A \otimes dz^A = g^{(1)}_{S^{n-2}}$ (12.61)

and writing χ instead of ξ:

$$g = c\,dt \otimes c\,dt - R^2 \sinh^2(ct/R) \underbrace{\left\{ d\chi \otimes d\chi + \sinh^2(\chi)\, g^{(1)}_{S^{n-2}} \right\}}_{\hat{g}_{K=-1}} \qquad (12.62)$$

$$\left.\begin{aligned} &= c\,dt \otimes c\,dt - a^2(t)\, \hat{g}_{K=-1} \\ &a(t) = R \sinh(ct/R) \end{aligned}\right\} \qquad (12.63)$$

This is just case 3, i.e. (11.49), again with

$$R = \left(\frac{3}{\Lambda}\right)^{1/2} \qquad (12.63)$$

Extra care: The static form

Locally there is also a static coordinate system on the de Sitter manifold, i.e. in which $g_{\alpha\beta}$ are t-independent and $g_{0a} = 0$, for $a = 1, 2, 3$. It is given by

$$X^0 = (R^2 - r^2)^{1/2} \sinh(ct/R) \tag{12.64a}$$

$$X^1 = (R^2 - r^2)^{1/2} \cosh(ct/R) \tag{12.64b}$$

$$X^A = r Z^A \quad , \quad 2 \leq A \leq n \tag{12.64c}$$

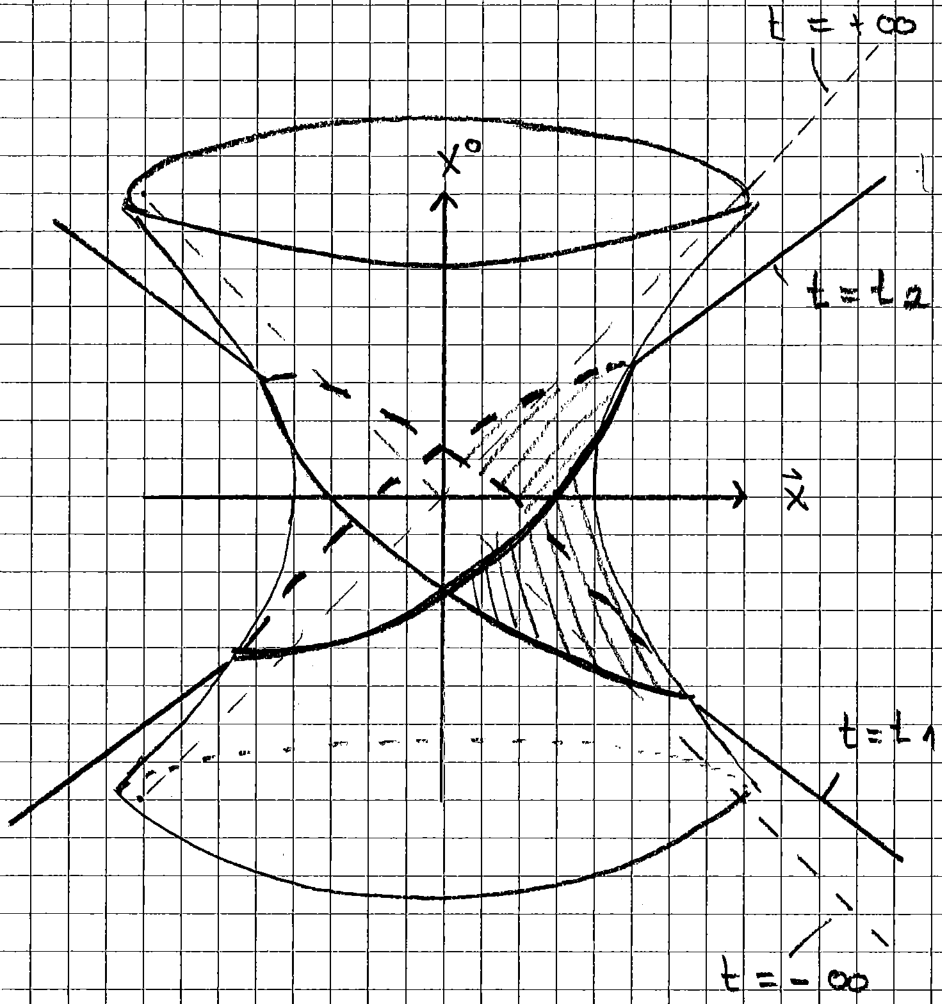
where
$$r^2 := \sum_{A=2}^{n} X^A X^A \tag{12.64d}$$

and
$$1 = \sum_{A=2}^{n} Z^A \cdot Z^A \tag{12.64e}$$

Again this is a non surjective isometric embedding with the hypersurfaces of constant t being given by the intersections of H^n_R with the hyperplanes of constant

$$\frac{X^0}{X^1} = \tanh(ct/R) \tag{12.65}$$

which are hyperplanes through the origin tilted by less than 45°.



(12.66)

This coordinate system covers only the right-wedge-cut-region of H^n_R between the planes $X^0 = X^1 \Leftrightarrow$ $\tanh(ct/R) = 1 \Leftrightarrow t \to \infty$ and $X^0 = -X^1 \Leftrightarrow \tanh(ct/R) = -1 \Leftrightarrow$ $t \to -\infty$. Note that from (12.64)

$$X^1 > |X^0| > 0 \tag{12.67}$$

and $\quad r^2 < R^2$

Now, using $f(r) := (R^2 - r^2)^{1/2}$, we get

$$dX^0 = \frac{f(r)}{R} \cosh(ct/R)\, c\,dt + \sinh(ct/R)\, df \qquad (12.68)$$

$$dX^1 = \frac{f(r)}{R} \sinh(ct/R)\, c\,dt + \cosh(ct/R)\, df \qquad (12.69)$$

$$\rightsquigarrow \quad dX^0 \otimes dX^0 - dX^1 \otimes dX^1$$

$$= \frac{f^2(r)}{R^2}\, c\,dt \otimes c\,dt - df \otimes df$$

$$= \frac{f^2(r)}{R^2}\, c\,dt \otimes c\,dt - f^{-2}(r)\, r^2\, dr \otimes dr \qquad (12.70)$$

Also

$$dX^A = Z^A\, dr + r\, dZ^A \qquad (12.71)$$

and since $\sum Z^A dZ^A = 0$ due to (12.64e)

$$\sum_{A=2}^{n} dX^A \otimes dX^A = dr \otimes dr + r^2 \sum_{A=2}^{n} dZ^A \otimes dZ^A \qquad (12.72)$$

$$\Rightarrow dX^0 \otimes dX^0 - dX^1 \otimes dX^1 - \sum_{A=2}^{n} dX^A \otimes dX^A$$

$$= \left(1 - \frac{r^2}{R^2}\right) c\,dt \otimes c\,dt - \left(\frac{r^2}{f^2} + 1\right) dr \otimes dr$$

$$- r^2 \sum dZ^A \otimes dZ^A \qquad (12.73)$$

Using $\sum_{A=2}^{n} dz^A \otimes dz^A = g^{(1)}_{S^{n-2}}$ (12.74)

we get

$$g = \left(1 - \frac{r^2}{R^2}\right) c\,dt \otimes c\,dt$$

$$- \left(1 - \frac{r^2}{R^2}\right)^{-1} dr \otimes dr$$

$$- r^2 g^{(1)}_{S^{n-2}} \qquad (12.75)$$

This is a static metric with Killing field

$$K = \frac{\partial}{\partial t} \qquad (12.76)$$

BUT : $\partial/\partial t$ is **not** geodesic

In fact, the acceleration of $\partial/\partial t$ diverges as $r \nearrow R$. The hypersurfaces $r = R$ are lightlike ($\partial/\partial t$ is lightlike there) and correspond to a Killing horizon, here also known as cosmological horizon. Since $\partial/\partial t$ is not geodesic, the static form is not a cosmological model. Galaxies will not move along $\partial/\partial t$.

anti-de Sitter space

The construction is similar to the de Sitter case, but instead of embedding a hyperboloid in $(n+1)$-dimensional Minkowski space $\mathbb{R}^{(1,n)}$, we take $\mathbb{R}^{(2,n-1)}$, i.e. as manifold $\mathbb{R}^{n+1}$, but an inner product of signature $(2, n-1)$:

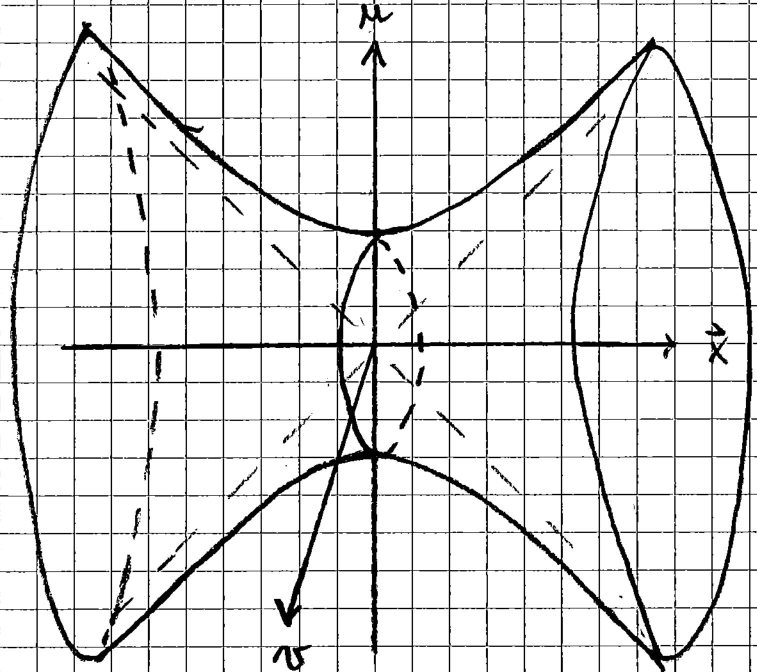
$$\eta = du \otimes du + dv \otimes dv - \sum_{a=1}^{n-1} dx^a \otimes dx^a \tag{12.77}$$

In that space we define

$$(adS)^n_R := \{ X \in \mathbb{R}^{(2,n-1)} : \eta(X,X) = R^2 \} \tag{12.78}$$

with induced metric

$$g = \eta \big|_{(adS)^n_R} \tag{12.79}$$



(12.80)

This looks like the picture (12.9) for de Sitter, rotated by $90°$. However, note that here u, v are both time-axes, i.e. enter η with a positive sign. Also, in de Sitter space we have $\eta(x, x) = -R^2$, here we have $\eta(x, x) = +R^2$. In de Sitter this condition removes one spatial dimension, here it removes one (of the two) time dimensions, leaving in each case a Lorentzian manifold.

Case 5: Open cosmology in $(adS)^n_R$

$$u = R \sin(ct/R) \cosh(\chi) \tag{12.81a}$$

$$v = R \cos(ct/R) \tag{12.81b}$$

$$X^A = R \sin(ct/R) \sinh(\chi) Z^A \tag{12.81c}$$

for $2 \leq A \leq n$

and the Z^A are again coordinates on the $(n-2)$ sphere, i.e. satisfy

$$\sum_{A=2}^{n} Z^A Z^A = 1 \tag{12.81d}$$

Note the similarity of (12.81) with (12.52). Note that (12.81) indeed satisfies

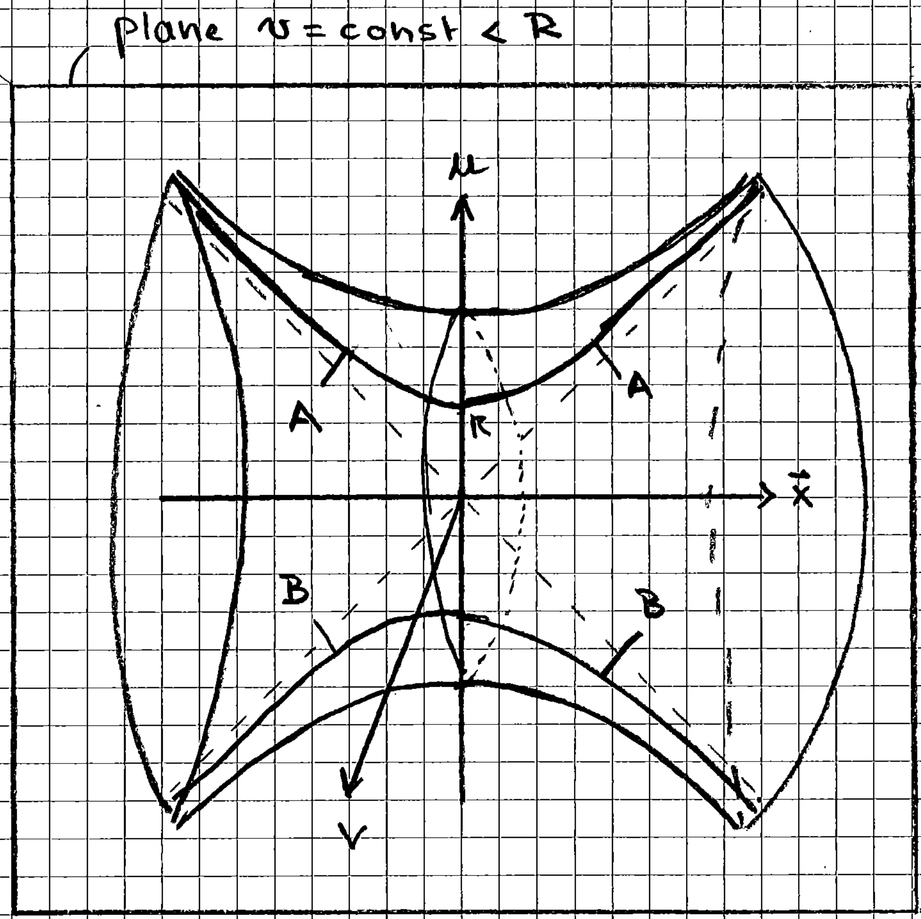
$$u^2 + v^2 - \sum_{A=2}^{n} (X^A)^2 = R^2 \tag{12.82}$$

identically, so that we have an embedding into $(adS)^n_R$. The hypersurfaces of constant time t are the intersections of the hyperplanes $v = \text{const.}$ with $(adS)^n_R$

Since $v \leq R$ we see from

$$u^2 - \sum_{A=2}^{n} (X^A)^2 = R^2 - v^2 \geq 0 \qquad (12.83)$$

that the intersections are – as in case 3 – spacelike hyperbola of constant negative sectional curvature. But also note that – in contrast to case 3 – these planes $v = \text{const.}$ move from $v = R$ for $t = 0$ (the "Big-Bang") to $v = 0$ for $t = (\pi/2)(R/c)$ (which corresponds to the moment t_{max} in (11.73) of maximal extent) and then further to $v = -R$ for $t = \pi \cdot R/c$ (which corresponds to the "Big-Crunch" and $t = \pi R/c = t_*$ to the lifetime (11.74)). Hence the planes $v = \text{const}$ move "inwards" within $v \in [-R, R]$ whereas in case 3 the planes $X^1 = \text{const}$ move from $X^1 = R$ for $t = 0$ "outwards" towards larger values of X^1.



(12.84)

The plane $v = \text{const} < R$ intersects adS^n_R in the two hyperbola A and B, but only A is parametrised by (12.81) since (12.81a) lets u raise from $u = 0$ at $t = 0$ to $u = R\cosh(x)$ at $t = (\pi/2)(R/c)$ and back to $u = 0$ at $t = \pi R/c$. Hence $u \geq 0$ for $t \in [0, \pi R/c]$. Hence this model is similar to the Milne universe (case 4) but in adS^n_R rather than Minkowski space.

From 12.81 we get

$$du = \cos(ct/R)\cosh(\chi)\, c\,dt$$
$$+ R\sin(ct/R)\sinh(\chi)\, d\chi \quad (12.85)$$

$$dv = -\sin(ct/R)\, c\,dt \quad (12.86)$$

$$dX^A = \cos(ct/R)\sinh(\chi) z^A\, c\,dt$$
$$+ R\sin(ct/R)\cosh(\chi) z^A\, d\chi$$
$$+ R\sin(ct/R)\sinh(\chi)\, dz^A \quad (12.87)$$

Since $\sum z^A z^A = 1 \curvearrowright \sum z^A dz^A = 0.$ (12.88)
Hence

$$\sum_{A=2}^{n} dX^A \otimes dX^A$$

$$= \cos^2(ct/R)\sinh^2(\chi)\, c\,dt \otimes c\,dt$$
$$+ R^2\sin^2(ct/R)\cosh^2(\chi)\, d\chi \otimes d\chi$$
$$+ R\sin(ct/R)\cos(ct/R)\sinh(\chi)\cosh(\chi)$$
$$\times (c\,dt \otimes d\chi + d\chi \otimes c\,dt)$$
$$+ R^2\sin^2(ct/R)\sinh^2(\chi)\sum_{A=2}^{n} dz^A \otimes dz^A \quad (12.89)$$

The $dt \otimes dx$ - cross terms from $du \otimes du$ cancel against that from $\Sigma\, dx^A \otimes dx^A$. The $cdt \otimes cdt$ terms from $du \otimes du$, $dv \otimes dv$, and $\Sigma\, dx^A \otimes dx^A$ combine to $cdt \otimes cdt$. And the $d\chi \otimes d\chi$ terms from $du \otimes du$ and $\Sigma\, dx^A \otimes dx^A$ combine to $R^2 \sin^2(ct/R)\, d\chi \otimes d\chi$. All this is very similar to the calculation for case 3 on pages (12.17-18). Here we get

$$du \otimes du + dv \otimes dv - \sum_{A=2}^{n} dx^A \otimes dx^A$$

$$= cdt \otimes cdt - R^2 \sin^2(ct/R) \Big\{ d\chi \otimes d\chi + \sinh^2(\chi) \sum_{A=2}^{n} dz^A \otimes dz^A \Big\} \qquad (12.90)$$

Again, since $\Sigma\, dz^A \otimes dz^A$ is the metric on the unit $(n-2)$ sphere we have

$$\hat{g}_{k=-1} = d\chi \otimes d\chi + \sinh^2(\chi)\, g^{(1)}_{S^{n-2}} \qquad (12.91)$$

and

$$g = du \otimes du + dv \otimes dv - \sum_{A=2}^{n} dx^A \otimes dx^A$$

$$= cdt \otimes cdt - a^2(t)\, \hat{g}_{k=-1} \qquad (12.92a)$$

$$a(t) = R \sin(ct/R). \qquad (12.92b)$$

That is, we get case 5 (11.71) with

$$R = \left(\frac{3}{|\Lambda|}\right)^{1/2} \qquad (12.93)$$

Like in the de Sitter case we can use (12.92) to calculate the Riemann-tensor via (4.78) in orthonormal basis $\theta^0 = c\,dt$, $\theta^a = a(t)\,\hat{\theta}^a$:

$$R_{0a0b} = \frac{\ddot{a}}{c^2 a}\,\delta_{ab} \qquad (12.94a)$$

$$R_{abab} = -\left(k + \frac{\dot{a}^2}{c^2}\right)/a^2 \quad (a \neq b) \qquad (12.94b)$$

$$\left.\begin{aligned} a(t) &= R\sin(ct/R) \\ \dot{a}(t) &= c\cos(ct/R) \\ \ddot{a}(t) &= -(c^2/R)\sin(ct/R) \end{aligned}\right\} \qquad (12.95)$$

$$\Rightarrow R_{0a0b} = -\frac{1}{R^2}\,\delta_{ab} \qquad (12.96)$$

$$R_{abab} = -\frac{(-1+\cos^2(ct/R))}{R^2\sin^2(ct/R)} = \frac{1}{R^2} \qquad (12.97)$$

Hence, since $\delta_{ab} = -g_{ab}$, $g_{00} = 1$,

$$R_{0a0b} = \frac{1}{R^2}\left(g_{00}\,g_{ab} - g_{0a}\,g_{0b}\right) \qquad (12.98)$$

$$R_{abab} = \frac{1}{R^2}\left(g_{aa}\,g_{bb} - g_{ab}\,g_{ab}\right) \qquad (12.99)$$

or

$$R_{\alpha\beta\mu\nu} = \frac{1}{R^2}\left(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}\right) \qquad (12.100)$$

Note the sign difference to (12.31).
[Had we chosen a "mostly plus" sign-convention the sign in (12.31) would be positive and negative here.]

Like for de Sitter we now get for anti-de Sitter the Ricci- and Einstein tensors:

$$R_{\alpha\beta} = R^{\lambda}{}_{\alpha\lambda\beta} = \frac{n-1}{R^2} g_{\alpha\beta} \qquad (12.101)$$

$$R = \frac{n(n-1)}{R^2} \qquad \text{(Ricci scalar left, radius of ads right)} \qquad (12.102)$$

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$$

$$= \left((n-1) - \frac{1}{2} n(n-1)\right) R^{-2} g_{\alpha\beta}$$

$$= -\frac{1}{2}(n-1)(n-2)\frac{1}{R^2} g_{\alpha\beta} \qquad (12.103)$$

$$\Rightarrow G_{\alpha\beta} - \Lambda g_{\alpha\beta} = 0 \qquad (12.104)$$

with

$$\Lambda = -\frac{(n-1)(n-2)}{2R^2} \qquad (12.105)$$

or

$$R = \left(\frac{(n-1)(n-2)}{2|\Lambda|}\right)^{1/2} \qquad (12.106)$$