Lecture 2: The Newtonian Friedmann Equations

We consider a spatially isotropic velocity field

$$\vec{v}(t,\vec{x}) = A(t)\cdot\vec{x} \qquad (2.1)$$

Spatial isotropy

$$\left.\begin{aligned} &v(t,R\vec{x}) = R\vec{v}(t,\vec{x}) \\ &\forall R \in SO(3) \end{aligned}\right\} \qquad (2.2)$$

$$\Leftrightarrow \quad R\,A(t)\,R^T = A(t)$$

$$\Leftrightarrow \quad A(t) = f(t)\,\mathrm{id}_{\mathbb{R}^3} \qquad (2.3)$$

$$\text{or} \quad A^a{}_b(t) = f(t)\,\delta^a{}_b \qquad (2.4)$$

Hence $\vec{v}(t,\vec{x}) = f(t)\,\vec{x}$ (2.5)

Mass conservation (1.49) now reads

$$\frac{\dot{\varrho}}{\varrho} = -3f(t) \qquad (2.6)$$

and momentum conservation (1.50) reads

$$(\dot{f} + f^2)\,\vec{x} = -\vec{\nabla}\phi \qquad (2.7)$$

Taking the divergence of that and using the field equation with cosmological constant

$$\Delta\phi + \Lambda = 4\pi G \varrho \qquad (2.8)$$

we obtain

$$3(\dot{f} + f^2) = -\Delta\phi = \Lambda - 4\pi G\varrho \qquad (2.9)$$

$$\Rightarrow \quad \dot{f} + f^2 = -\frac{4\pi}{3} G\varrho + \frac{\Lambda}{3} \qquad (2.10)$$

We now define $R(t)$ through

$$f(t) = \dot{R}(t)/R(t) \qquad (2.11)$$

so that

$$R(t) = c\,\exp\left(\int_0^t dt'\, f(t')\right) \qquad (2.12)$$

Then the flow of velocity vector field $\vec{V}(t,\vec{x}) = f(t)\,\vec{x}$ is given by

$$\dot{\vec{x}}(t) = \vec{V}(t,\vec{x}(t)) = \frac{\dot{R}(t)}{R(t)}\,\vec{x}(t) \qquad (2.13)$$

$$\Rightarrow \quad \vec{X}(t) = R(t)\,\vec{n} \tag{2.14}$$

where
$$\vec{n} = \frac{\vec{X}}{\|\vec{X}\|} \tag{2.15}$$

The mass-conservation equation (2.6) now reads

$$\dot{\varrho}/\varrho = -3\,\dot{R}(t)/R(t) \tag{2.16}$$

and integrates to $\varrho(t)\,R^3(t) = \text{const.}$,
or

$$\boxed{M := \frac{4\pi}{3}\varrho R^3 = \text{const.}} \tag{2.17}$$

And, finally, the field equation (2.10) also simplifies, since

$$\dot{f} + f^2 = (\dot{R}/R)^{\dot{}} + (\dot{R}/R)^2 = \frac{\ddot{R}}{R} \tag{2.18}$$

Hence (2.10) now reads

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3} G\varrho + \frac{\Lambda}{3}$$

$$= -G\frac{M}{R^3} + \frac{\Lambda}{3} \tag{2.19}$$

or

$$\boxed{\ddot{R} = -G\frac{M}{R^2} + \frac{\Lambda}{3}R} \tag{2.20}$$

Multiplikation of (2.20) by $\dot{R}$ gives a first integral

$$\frac{1}{2}\dot{R}^2 = G\frac{M}{R} + \frac{\Lambda}{6}R^2 - \frac{k}{2} \qquad (2.21)$$

or

$$\boxed{\left(\frac{\dot{R}}{R}\right)^2 = \frac{2GM}{R^3} - \frac{k}{R^2} + \frac{\Lambda}{3}} \qquad (2.22)$$

The framed equations (2.17), (2.20) and (2.22) are called the <u>Friedmann-Equations</u> – after Alexander Friedmann who derived them from GR in 1922.

Note that according to our derivation $R(t)$ has no direct geometric meaning (as it will acquire in GR). Rather its significance and interpretation lies in

$$\vec{v}(t,\vec{x}) = \frac{\dot{R}(t)}{R(t)}\,\vec{x}\,. \qquad (2.23)$$

i.e. the velocity with which each observer sees the "cosmological fluid" around him/her in a radial motion. Recall that each observer sees the <u>same</u> relative velocity field (comp. page (1.9-10.)).

In GR we do get the same equations in case of a "matter dominated" universe, i.e. where $\rho \gg p/c^2$. The constants k and Λ are then proportional to the curvature and cosm. constant. We will see that.

Let $t = t_0$ be a distinguished time, the "now". Set

$$H_0 := \frac{\dot{R}(t_0)}{R(t_0)} \qquad (2.24)$$

$$= \text{Hubble Parameter}$$

so that

$$\vec{V}(t_0, \vec{X}) = H_0 \vec{X} \qquad (2.25)$$

(Velocity - distance relation)

We define the dimensionless parameters

$$\Omega_m := H_0^{-2} \frac{2GM}{[R(t_0)]^3} \qquad (2.26a)$$

$$\Omega_\Lambda := H_0^{-2} \frac{\Lambda}{3} \qquad (2.26b)$$

$$\Omega_k := -H_0^{-2} \frac{k}{[R(t_0)]^2} \qquad (2.26c)$$

The Ω's are called the "cosmological parameters". They quantify how much matter (M, or ϱ), cosm. constant (Λ) and constant k (later to be seen to be due to spatial curvature) contribute to "driving the universe". Note that due to (2.17)

$$\Omega_m := H_0^{-2} \frac{2G}{R_0^3} \frac{4\pi}{3} \varrho_0 R_0^3$$

$$= H_0^{-2} \frac{8\pi G}{3} \varrho_0 \qquad (2.27)$$

i.e. Ω_m is a direct measure for the current matter density

The traditional phrases the Ω's are called by are "Omega-Matter" (Ω_m), "Omega-Lambda (Ω_Λ), and "Omega-Curvature" (Ω_k).

Now, using (2.26), $(2.22)\big|_{t=t_0}$ reads

$$\boxed{\Omega_m + \Omega_\Lambda + \Omega_k = 1} \qquad (2.28)$$

"cosmological triangle"

If we evaluate (2.20) at $t = t_0$ and divide by $H_0^2 R_0$, we get

$$\frac{\ddot{R}_0}{R_0 H_0^2} = \Omega_\Lambda - \frac{1}{2}\Omega_m \tag{2.29}$$

hence

$$\ddot{R}(t_0) \begin{cases} > 0 & \text{for } \Omega_\Lambda > \frac{1}{2}\Omega_m \\ < 0 & \text{for } \Omega_\Lambda < \frac{1}{2}\Omega_m \end{cases} \tag{2.30}$$

If $H(t) = \dot{R}(t)/R(t) > 0$ we say the "universe is expanding" or "contracting" if $H(t) < 0$. If in an expanding universe is said to be "accelerated expanding" if $\ddot{R}(t) > 0$.

We now rewrite the equation of motion for R, i.e. (2.22), in a dimensionless form using the Ω's. First we define the dimensionless dependent variable

$$X(t) := \frac{R(t)}{R(t_0)} \tag{2.31}$$

and the dimensionless independent variable

$$\lambda := H_0 \, t = \frac{\dot{R}(t_0)}{R(t_0)} \, t \, . \qquad (2.32)$$

Then

$$\frac{H(t)}{H_0} = \frac{\dot{R}/R}{\dot{R}_0/R_0} = \frac{1}{X} \, \frac{\dot{R}}{\dot{R}_0}$$

$$= \frac{1}{X} \, \frac{\dot{R}/R_0}{\dot{R}_0/R_0}$$

$$= \frac{1}{X} \, \frac{1}{H_0} \, \frac{dX}{dt}$$

$$= \frac{1}{X} \, \frac{dX}{d\lambda} \qquad (2.33)$$

and $(2.22) \times H_0^{-2}$ turns into

$$\frac{H^2}{H_0^2} = \frac{1}{X^2} \left(\frac{dX}{d\lambda} \right)^2$$

$$= X^{-3} \, \Omega_m + X^{-2} \, \Omega_k + \Omega_\Lambda \qquad (2.34)$$

$$\boxed{\left(\frac{dX}{d\lambda} \right)^2 - \frac{1}{X} \, \Omega_m - X^2 \, \Omega_\Lambda = \Omega_k} \qquad (2.35)$$

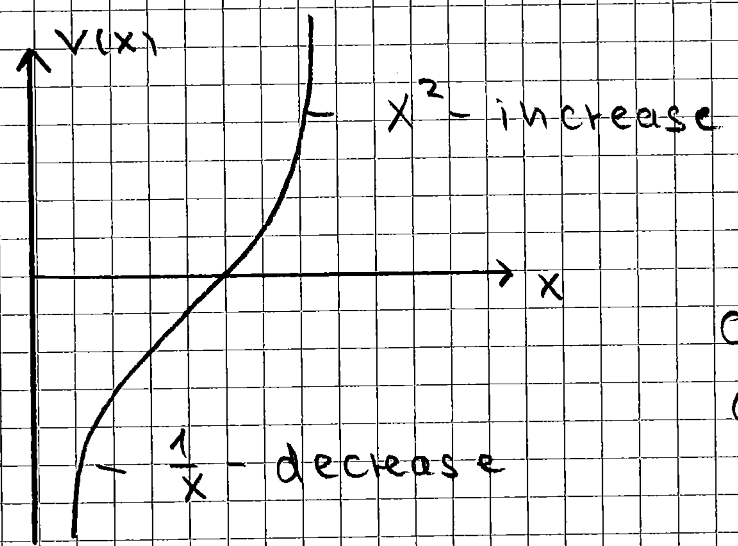
This may be compared to the equation for a particle of mass = 2 moving in a potential

$$V(x) = -\left(\frac{\Omega m}{x} + x^2 \Omega \Lambda\right) \qquad (2.36)$$

with overall energy $E = \Omega k = 1 - \Omega m - \Omega \Lambda$. Here $\Omega m > 0$ but $\Omega \Lambda$ may have either sign.

Discussion of $V(x)$

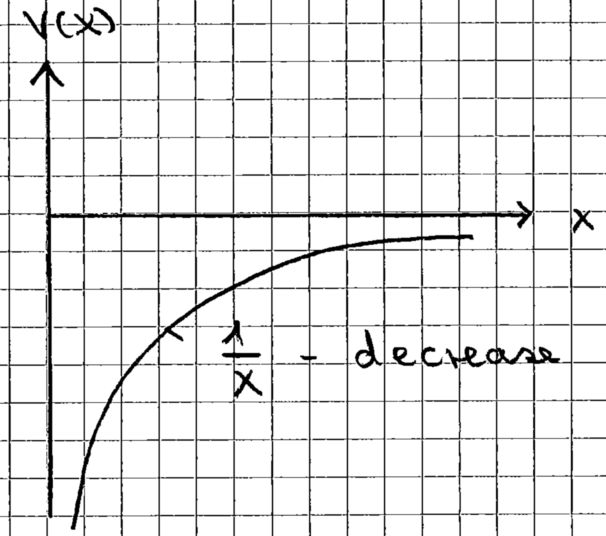
1. Case: $\Omega \Lambda < 0$, i.e. $\Lambda < 0$



(2.37)

always collapsing

2. Case : $\Omega_\Lambda = 0$, i.e. $\Lambda = 0$



(2.38)

2a) $$E < 0 \iff \Omega_K < 0 \iff \Omega_m > 1 \qquad (2.39)$$

By (2.27):

$$\Omega_m = H_0^{-2} \frac{8\pi G}{3} \rho_0 \qquad (2.40)$$

hence $\Omega_m > 1 \iff$

$$\rho > \rho_{crit} := \frac{3}{8\pi G} H_0^2 \qquad (2.41)$$

In cosmology H_0 is often given in units of $100 \, km \cdot s^{-1} \cdot (Mpc)^{-1}$

$$H_0 = h_0 \cdot 100 \, \frac{km}{s} \cdot (Mpc)^{-1} \qquad (2.42)$$

$$Mpc \cong 3.1 \times 10^{19}\ km$$

$$\rho_{crit} = \frac{3}{8\pi G} \frac{10^4\ km^2 \cdot s^{-2}}{(3.1 \times 10^{19}\ km)^2} h_0^2$$

$$= h_0^2 \cdot 1.87 \cdot 10^{-26}\ kg\, m^{-3} \quad (2.43a)$$

$$= h_0^2 \cdot 1.87 \cdot 10^{-29}\ g \cdot cm^{-3} \quad (2.43b)$$

The currently best value for h_0 is about 0.71

Since $E = 1 - \Omega_m$ we have a collapsing universe, $E < 0$, if

$$\rho_0 > \rho_{crit} \quad (2.44)$$

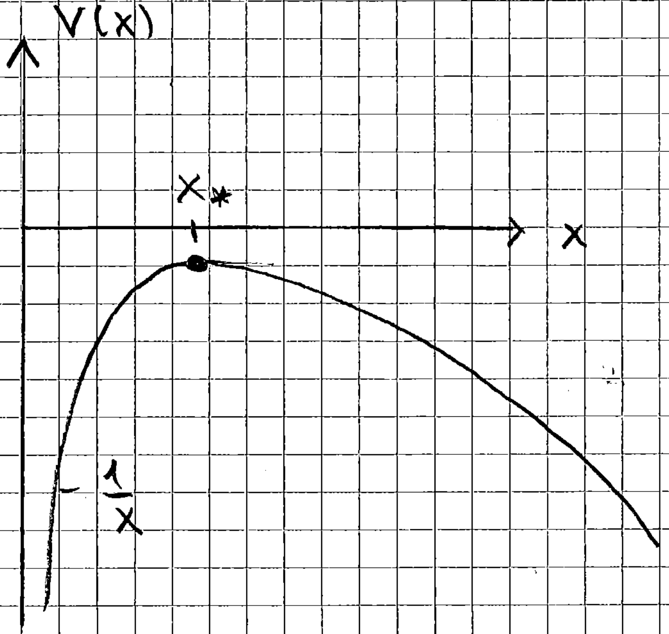
2b) $E > 0$

$\Leftrightarrow \quad \Omega_m < 1$

$\Leftrightarrow \quad \rho_0 < \rho_{crit} \quad (2.45)$

ever expanding universe.

3. Case: $\Omega_\Lambda > 0$, i.e. $\Lambda > 0$



(2.46)

$$V(x) = -x^2\,\Omega_\Lambda - \frac{1}{x}\,\Omega_m \qquad (2.47)$$

$$V'(x) = -2x\,\Omega_\Lambda + \frac{1}{x^2}\,\Omega_m \qquad (2.48)$$

$$V'(x) = 0 \iff$$

$$x = x_* := \left(\Omega_m / 2\Omega_\Lambda\right)^{1/3} \qquad (2.50)$$

(Maximum)

Value of potential at maximum

$$V(x_*) = -\left[\left(\frac{\Omega_m}{2\Omega_\Lambda}\right)^{2/3}\Omega_\Lambda + \left(\frac{2\Omega_\Lambda}{\Omega_m}\right)^{1/3}\Omega_m\right]$$

$$= -\underbrace{\left(2^{-2/3} + 2^{1/3}\right)}_{2^{1/3}\left(1+\frac{1}{2}\right) = 3/2^{2/3}}\left(\Omega_m^2\,\Omega_\Lambda\right)^{1/3}$$

Hence

$$V(X_*) = -\frac{3}{2}\left(2\,\Omega_m^2\,\Omega_\Lambda\right)^{1/3} \qquad (2.51)$$

Case 3a)

$$E > V(X_*) \qquad (2.52)$$

The "particle" climbs over the potential maximum and we have either eternal expansion (if particle approaches maximum from left, i.e if universe is expanding) or continued collapse (if it approaches maximum from right).

Equation (2.52) is equivalent to

$$\Omega_K = (1-\Omega_m-\Omega_\Lambda)$$

$$> -\frac{3}{2}\left(2\,\Omega_m^2\,\Omega_\Lambda\right)^{1/3} \qquad (2.53)$$

or

$$(1-\Omega_m-\Omega_\Lambda)^3 > \frac{27}{4}\,\Omega_m^2\,\Omega_\Lambda \qquad (2.54)$$

Case 3b)

$$E < V(X_*) \tag{2.55}$$

The "particle" bounces back from the potential mountain, either to recollapse, (if it came from left, expanding), or to reexpand (if it came from right, contracting). Equation (2.55) is equivalent to

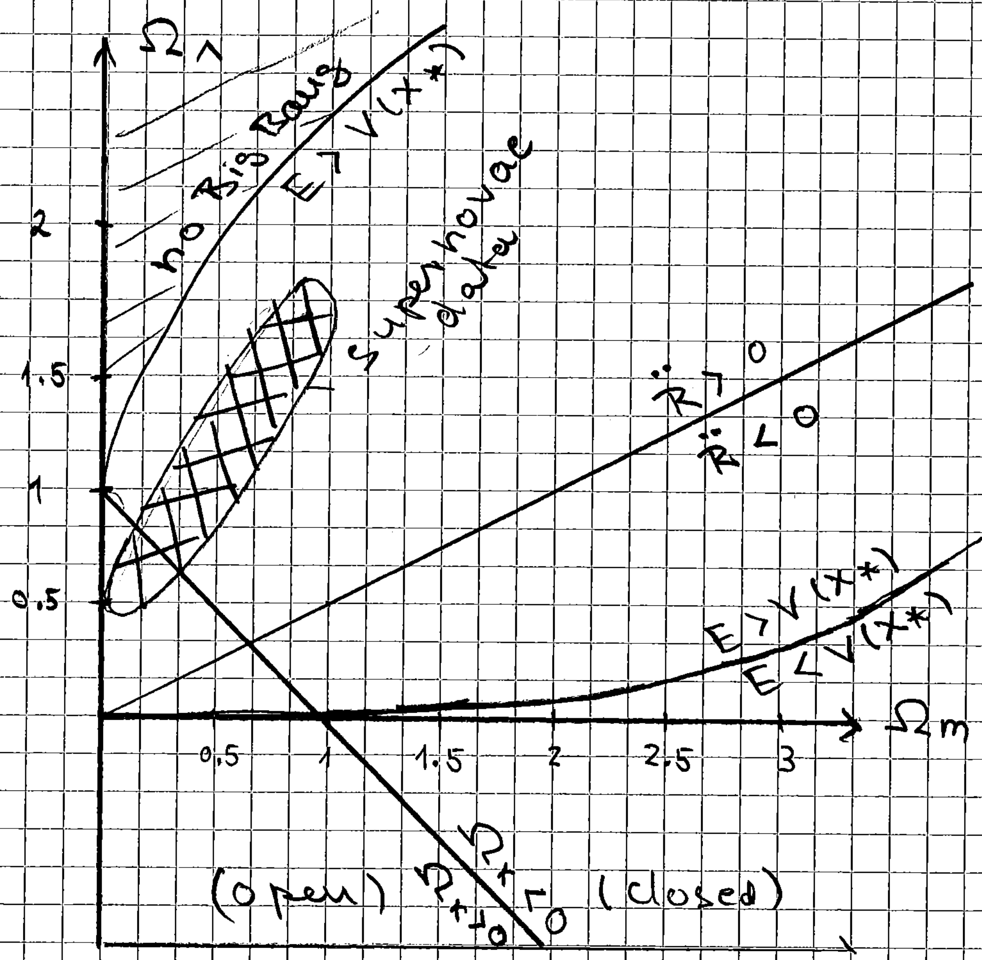
$$(1 - \Omega_m - \Omega_\Lambda)^3 < \frac{27}{4}\,\Omega_m^2\,\Omega_\Lambda \tag{2.56}$$

Equations (2.54) and (2.56) define regions in the $(\Omega_m, \Omega_\Lambda)$-plane, the separating curve between which is given by

$$(1 - \Omega_m - \Omega_\Lambda)^3 = \frac{27}{4}\,\Omega_m^2\,\Omega_\Lambda \tag{2.57}$$

$$\Rightarrow \quad \Omega_m = 0 \Rightarrow \Omega_\Lambda = 1 \tag{2.58a}$$

$$\Omega_\Lambda = 0 \Rightarrow \Omega_m = 1 \tag{2.58b}$$



Taken together with Galaxy count (giving $\Omega_m \in [0.3 - 0.42]$

and CMB data (giving $\Omega_k \cong 0$) yields.

Current best values are

$$\Omega_m = 0.31 \tag{2.58a}$$

$$\Omega_\Lambda = 0.69 \tag{2.58b}$$

$$\Omega_k \cong 0 \tag{2.58c}$$

With $\underbrace{\Omega_m = \Omega_b}_{\text{baryonic}} + \underbrace{\Omega_c}_{\text{dark matter}}$ (2.59)

$$\Omega_m = \Omega_b + \Omega_c$$

$\Leftrightarrow$

$$0.31 = 0.0486 + 0.2589 \quad (2.60)$$

$$\frac{\Omega_b}{\Omega_m} = \frac{15,7}{100} = \frac{1}{6.38} \quad (2.61)$$

$$\frac{\Omega_b}{\Omega_m + \Omega_\Lambda} = \Omega_b = \frac{4.86}{100} = \frac{1}{20.1} \quad (2.62)$$

(2.61) is called the "dark-matter-problem", i.e. why is Ω_b so much smaller than Ω_m and what is $\Omega_m - \Omega_b$ made of.

(2.62) is called the "dark-energy-problem; i.e. why is Ω_Λ of same magnitude – roughly – as Ω_m and what causes it.

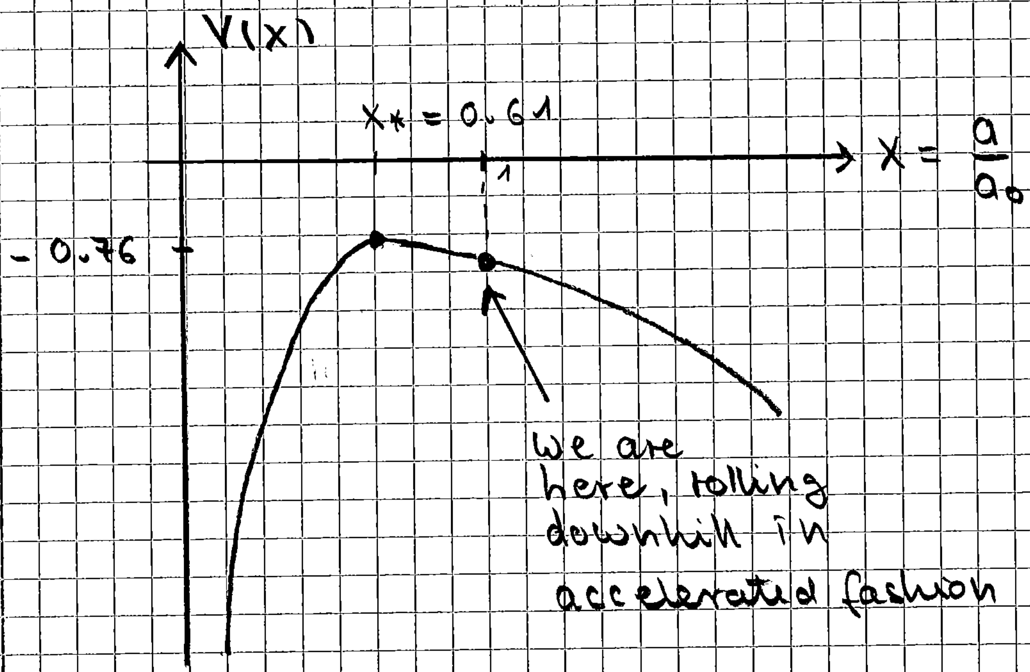
With (2.58 a-b) we get for our universe

$$x_* = \left(\frac{0.31}{2 \cdot 0.69}\right)^{1/3} = 0.61 \quad (2.63)$$

$$V(x_*) = -\frac{3}{2}\left(2\cdot(0.31)^2\cdot 0.69\right)^{1/3}$$

$$= -0.76 \qquad (2.64)$$

Whereas $E = \Omega_k \cong 0$ (2.65)



(2.66)

Acceleration parameter

$$q := \frac{\ddot{R}(t_0)\, R(t_0)}{[\dot{R}(t_0)]^2} = \frac{\ddot{R}(t_0)}{R(t_0)\, H_0^2}$$

$$= \Omega_\Lambda - \frac{1}{2}\Omega_m$$

$$\cong 0.54 \quad > 0 \qquad (2.67)$$

Cosmological redshift

$$z = \frac{\lambda_{\text{received}} - \lambda_{\text{emitted}}}{\lambda_{\text{emitted}}} \qquad (2.68)$$

We will see that

$$z + 1 = \frac{\lambda_{\text{received}}}{\lambda_{\text{emitted}}} = \frac{R(t_0)}{R(t_1)} \qquad (2.69)$$

t_0 = time of reception ("now")
t_1 = time of emission $< t_0$

So for $t_1 = t_*$ = time at which universe is on top of potential hill, have

$$z_* = \frac{1}{x_*} - 1 = 0.64 \qquad (2.70)$$

That means: Objects seen with redshifts greater than 0.64 are seen at a time when the universe was still climbing the hill and in a phase of decelerated expansion. At $z = z_* = 0.64$ this phase turned into a phase of ever-lasting accelerated expansion.