Lecture 3: Some recap from GR

We will eventually discuss the FLRW models within GR.

F : Alexander Friedman (1888-1925)
L : George Lemaître (1894-1966)
R : Howard Robertson (1903-1961)
W : Arthur Walker (1909-2001)

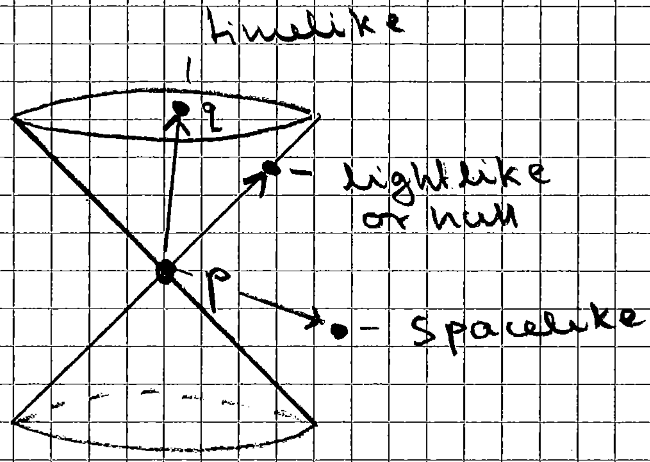
The idea of an expanding universe is essentially due to Lemaître (1927); the essential observations were made by Hubble in 1929.

Let us recall a few facts from Special and General Relativity (SR and GR).

In SR, Space time is Minkowski Space: A 4-d affine Space whose vector space has a symmetric, non-deg. bilinear form η of signature $(+,-,-,-)$. In diagonalising basis:

$$\eta(e_\alpha, e_\beta) = \eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1) \quad (3.1)$$

η gives Space time a causal structure



(3.2)

q is timelike, lightlike, or spacelike separated relative to p according to $\|q-p\|^2_\eta := \eta(q-p, q-p)$ being positive, zero or negative:

$$(q-p) = \begin{Bmatrix} \text{timelike} \\ \text{lightlike} \\ \text{spacelike} \end{Bmatrix} \Leftrightarrow$$

$$\eta(q-p, q-p) = \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases} \qquad (3.3)$$

q and p are <u>causally connected</u> iff $(q-p)$ is non-spacelike.

Two subsets $S_1, S_2 \subset M^4$ are <u>causally disjoint</u> if $\eta(p_1-p_2, p_1-p_2) < 0 \ \forall p_1 \in M_1, p_2 \in M_2$.

If $\lambda \longmapsto z(\lambda) \in M^4$ is a differentiable curve in space-time, it is called timelike, lightlike or spacelike according to $\dot{z}(\lambda)$ being that:

$$\eta_{\alpha\beta}\,\dot{z}^\alpha \dot{z}^\beta = \begin{cases} > 0 & \text{timelike} \\ = 0 & \text{lightlike} \\ < 0 & \text{spacelike} \end{cases} \tag{3.4}$$

For timelike or spacelike curves we can define a <u>length functional</u>:

$$\ell[z] := \int d\lambda \; |\eta(\dot{z},\dot{z})|^{1/2} \tag{3.5}$$

The so-called <u>energy functional</u> exists for all curves

$$E[z] := \int d\lambda \quad \eta(\dot{z},\dot{z}) \tag{3.6}$$

The length-functional is reparametrisation invariant, but the energy functional is not.

In GR we generalise η to a space-time dependent metric g

$$\begin{array}{lcl} SR & \longrightarrow & GR \\ \eta_{\alpha\beta} & \longrightarrow & g_{\alpha\beta}(x^\lambda) \end{array} \qquad (3.7)$$

The ten components $g_{\alpha\beta}(x^\lambda)$ obey 10 non-linear partial differential equations

$$G_{\alpha\beta} - \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta} \qquad (3.8)$$

$G_{\alpha\beta}$ ↓ Einstein Tensor

$T_{\alpha\beta}$ ↓ Energy-Momentum Tensor

$$T_{\alpha\beta} = \left(\begin{array}{c|c} W & c\vec{G}^T \\ \hline \frac{1}{c}\vec{S} & \Sigma_{ab} \end{array} \right) \qquad (3.9)$$

W = energy-density

$\vec{S}$ = energy-current density

$\vec{G}$ = momentum density

Σ_{ab} = momentum current density

$$\kappa = 8\pi G / c^4 \qquad (3.10)$$

We have

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \qquad (3.11)$$

$$R := g^{\alpha\beta} R_{\alpha\beta} \qquad (3.12)$$

where

$$R_{\alpha\beta} := R^{\lambda}{}_{\alpha\lambda\beta} \qquad (3.13)$$

↑ Ricci Tensor — ↑ Riemann Tensor

and

$$R^{\alpha}{}_{\beta\mu\nu} := \partial_\mu \Gamma^{\alpha}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\gamma} \Gamma^{\gamma}_{\nu\beta} - \Gamma^{\alpha}_{\nu\gamma} \Gamma^{\gamma}_{\mu\beta} \qquad (3.14)$$

and

$$\Gamma^{\lambda}_{\alpha\beta} := \frac{1}{2} g^{\lambda\gamma} \left(- g_{\alpha\beta,\gamma} + g_{\gamma\alpha,\beta} + g_{\beta\gamma,\alpha} \right) \qquad (3.15)$$

↑ Christoffel Symbols

Again we have an "energy functional"

$$E[Z] = \int d\lambda \; g_{\alpha\beta}(Z(\lambda))\, \dot{Z}^{\alpha}(\lambda)\, \dot{Z}^{\beta}(\lambda) \qquad (3.16)$$

the Euler-Lagrange equations of which are

$$\boxed{\ddot{Z}^{\alpha}(\lambda) + \Gamma^{\alpha}_{\beta\gamma}(Z(\lambda))\, \dot{Z}^{\beta}(\lambda)\, \dot{Z}^{\gamma}(\lambda) = 0} \qquad (3.17)$$

("geodesic equation")

Solutions obey ($k = \text{const.}$)

$$\boxed{g_{\alpha\beta}(Z(\lambda))\, \dot{Z}^{\alpha}(\lambda)\, \dot{Z}^{\beta}(\lambda) = k} \qquad (3.18)$$

with

$$k \begin{cases} > 0 & \text{timelike geodesic} \\ = 0 & \text{lightlike geodesic} \\ < 0 & \text{spacelike geodesic} \end{cases} \qquad (3.19)$$

Freely falling massive particles describe timelike geodesics, light rays lightlike geodesics.

If instead of λ we use a different parameter

$$\sigma = \sigma(\lambda) \tag{3.20}$$

then

$$\frac{d}{d\lambda} = \dot{\sigma} \frac{d}{d\sigma}, \tag{3.21a}$$

$$\frac{d^2}{d\lambda^2} = (\dot{\sigma})^2 \frac{d^2}{d\sigma^2} + \ddot{\sigma} \frac{d}{d\sigma}. \tag{3.21b}$$

(3.17) becomes ($z'^{\alpha} = dz^{\alpha}/d\sigma$ etc)

$$\ddot{\sigma} z'^{\alpha} + \dot{\sigma}^2 z''^{\alpha} + \dot{\sigma}^2 \Gamma^{\alpha}_{\beta\gamma} z'^{\beta} z'^{\gamma} = 0,$$

or equivalently

$$z''^{\alpha} + \Gamma^{\alpha}_{\beta\gamma} z'^{\beta} z'^{\gamma} = -\frac{\ddot{\sigma}}{\dot{\sigma}^2} z'^{\alpha}. \tag{3.22}$$

(3.18) becomes

$$g_{\alpha\beta} z'^{\alpha} z'^{\beta} = k / \dot{\sigma}^2. \tag{3.23}$$

Hence equation (3.17) remains invariant (with a zero on the r.h.s) under reparametrisations with

$$\ddot{\sigma} = 0 \iff \sigma(\lambda) = a\lambda + b \qquad a \in \mathbb{R} - \{0\},\ b \in \mathbb{R} \tag{3.24}$$

For those the r.h.s of (3.23) becomes

$$\tilde{k} := k / \dot{\sigma}^2 = k / a^2 \qquad (3.25)$$

Reparametrisations of the form (3.24) are called affine. Hence (3.17) and (3.18) are invariant under affine reparametrisations. For timelike $(k > 0)$ and spacelike $(k < 0)$ geodesics there is a preferred subset of parameters within the affine equivalence class, namely those for which $\tilde{k} = \pm 1$, i.e. $a = \sqrt{|k|}$. They are determined up to additive constants $\tilde{k} \to \tilde{k} + b$. These parameters are called "arc length". If λ was such a parameter, i.e. if

$$g_{\alpha\beta}(z(\lambda))\, \dot{z}^{\alpha}(\lambda)\, \dot{z}^{\beta}(\lambda) = \pm 1 \qquad (3.26)$$

Then

$$\ell\,[z] = \int_{\lambda_0}^{\lambda_1} d\lambda \; \left| g_{\alpha\beta}\, \dot{z}^{\alpha} \dot{z}^{\beta} \right|^{1/2}$$

$$= (\lambda_1 - \lambda_0) \qquad (3.27)$$

Hence λ corresponds to the arc-length as measured with respect to the metric g. For lightlike geodesics no such interpretation is possible and no preferred subset within the affine equivalence class of parameters exists. The parameters λ for which $z(\lambda)$ satisfies the geodesic equation (3.17) are called affine parameters

Summary: The geodesic equation for a curve $\lambda \mapsto z(\lambda)$ selects a preferred set of parameters, called the affine parameters. This set is acted upon by the affine group $\mathbb{R} \rtimes \mathbb{R}^*$, where $\mathbb{R}^* := \mathbb{R} - \{0\}$, with $(b, a) \in \mathbb{R} \rtimes \mathbb{R}^*$ acting on $\lambda \in \mathbb{R}$ like

$$(b, a)\lambda := a\lambda + b \qquad (3.28)$$

hence

$$(b', a')(b, a)\lambda = (b', a')(a\lambda + b)$$

$$= a'(a\lambda + b) + b' = a'a\lambda + a'b + b'$$

$$= (b' + a'b, a'a)\lambda \qquad (3.29)$$

The affine group multiplication is

$$(b', a')(b, a) = (b' + a'b,\ a'a) \qquad (3.30)$$

The action of $\mathbb{R} \rtimes \mathbb{R}^*$ on the set of allowed parameters is simply transitive, i.e. each allowed parameter can be obtained from a given one by exactly one (b, a).
For timelike and spacelike geodesics the set of affine parameters contains the subset of arc lengths, which are such that

$$| g_{\alpha\beta} \dot{z}^\alpha \dot{z}^\beta | = 1 \qquad (3.31)$$

For lightlike geodesics we have

$$g_{\alpha\beta} \dot{z}^\alpha \dot{z}^\beta = 0 \qquad (3.32)$$

and no concept of arc length exists. But the affine equivalence class of preferred parameters (within the much bigger set of all curve-parameters) does exist.

Given the $\Gamma^{\alpha}_{\beta\gamma}$, we can define covariant derivatives of tensor-fields:

Scalar : $\nabla_\mu \phi = \partial_\mu \phi$ (3.33)

Vector : $\nabla_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma^{\alpha}_{\mu\beta} V^\beta$ (3.34)

Co-Vektor : $\nabla_\mu V_\alpha = \partial_\mu V_\alpha - \Gamma^{\beta}_{\mu\alpha} V_\beta$ (3.35)

Tensor : $\nabla_\mu T^\alpha{}_\beta = \partial_\mu T^\alpha{}_\beta + \Gamma^{\alpha}_{\mu\gamma} T^\gamma{}_\beta - \Gamma^{\gamma}_{\mu\beta} T^\alpha{}_\gamma$ (3.36)

In general

$$\nabla_\mu T^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \cdots \beta_q} = \partial_\mu T^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \cdots \beta_q} + \sum_{i=1}^{p} \Gamma^{\alpha_i}_{\mu\lambda} T^{\alpha_1 \cdots \alpha_{i-1} \lambda \alpha_{i+1} \cdots \alpha_p}{}_{\beta_1 \cdots \beta_q} - \sum_{i=1}^{q} \Gamma^{\lambda}_{\mu\beta_i} T^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \cdots \beta_{i-1} \lambda \beta_{i+1} \cdots \beta_q}$$

(3.37)

It follows

$$\nabla_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - \Gamma_\mu{}^\gamma{}_\alpha g_{\gamma\beta} - \Gamma_\mu{}^\gamma{}_\beta g_{\alpha\gamma}$$

$$= 0 \qquad (3.38a)$$

Likewise

$$\nabla_\mu g^{\alpha\beta} = 0 \qquad (3.38b)$$

The metric is covariantly constant.

There is an identity

$$\nabla^\alpha G_{\alpha\beta} = 0 \qquad (3.38c)$$

(twice contracted 2nd Bianchi-Id.)

From Einstein Equation

$$G_{\alpha\beta} - \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta} \qquad (3.38d)$$

$$\rightsquigarrow \quad \nabla^\alpha T_{\alpha\beta} = 0 \qquad (3.38e)$$

(Integrability Equation)

The Lie-derivative of a tensor field T w.r.t a vector field X is

$$(L_X T)^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q}$$

$$= X^\mu \partial_\mu T^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q}$$

$$- \sum_{i=1}^{p} X^{\alpha_i}{}_{,\lambda} \, T^{\alpha_1 \cdots \alpha_{i-1} \lambda \alpha_{i+1} \cdots \alpha_p}_{\beta_1 \cdots \beta_q}$$

$$+ \sum_{i=1}^{q} X^{\lambda}{}_{,\beta_i} \, T^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_{i-1} \lambda \beta_{i+1} \cdots \beta_q} \qquad (3.39)$$

$$= X^\mu \nabla_\mu T^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q}$$

$$- \sum_{i=1}^{p} \nabla_\lambda X^{\alpha_i} \, T^{\alpha_1 \cdots \alpha_{i-1} \lambda \alpha_{i+1} \cdots \alpha_p}_{\beta_1 \cdots \beta_q}$$

$$+ \sum_{i=1}^{q} \nabla_{\beta_i} X^{\lambda} \, T^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_{i-1} \lambda \beta_{i+1} \cdots \beta_q} \qquad (3.40)$$

In the first set of expressions we set

$$X^\alpha{}_{,\beta} := \partial_\beta X^\alpha = \frac{\partial X^\alpha}{\partial x^\beta} . \qquad (3.41)$$

For example, the Lie derivative of the metric g is

$$(L_X g)_{\alpha\beta}$$

$$= X^\mu \partial_\mu g_{\alpha\beta}$$

$$- X^\lambda{}_{,\alpha} g_{\lambda\beta} - X^\lambda{}_{,\beta} g_{\alpha\lambda}$$

$$= X^\mu \nabla_\mu g_{\alpha\beta}$$

$$- (\nabla_\alpha X^\lambda) g_{\lambda\beta} - (\nabla_\beta X^\lambda) g_{\alpha\lambda}$$

$$= - \nabla_\alpha X_\beta - \nabla_\beta X_\alpha \qquad (3.42)$$

<u>Definition</u> A vector field X is called a Killing field for the metric g iff

$$L_X g = 0 \qquad (3.43)$$

<u>Lemma</u> The set of Killing fields for a given metric g is a vector space

Proof: $L_{X_1} g = L_{X_2} g = 0$

$$\Rightarrow \quad L_{X_1 + a X_2} g = 0 \quad , a \in \mathbb{R}.$$

∎

<u>Theorem</u>: Let (M, g) be a metric manifold of dimension $n \geq 2$. Metric means that g is a field of symmetric non-degenerate bilinear forms of any signature. Then the vector space of Killing fields is at most of dimension $\frac{1}{2}n(n+1)$.

Proof: We use the Killing equation

$$\boxed{\nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0} \qquad (3.43)$$

as well as the curvature relation

$$\nabla_\alpha \nabla_\beta X_\lambda - \nabla_\beta \nabla_\alpha X_\lambda = -R^\sigma{}_{\lambda\alpha\beta} X_\sigma \qquad (3.44)$$

to prove a relation that allows to express the 2nd derivatives of X in terms of X itself. This relation is the following identity

$$\boxed{\nabla_\alpha \nabla_\beta X_\gamma = R^\lambda{}_{\alpha\beta\gamma} X_\lambda} \qquad (3.45)$$

Hence all derivatives of $X_\alpha(p)$, $p \in M$, are, in view of (3.43, 45), determined by the n values $X_\alpha(p)$ and the $\frac{1}{2}n(n-1)$ values $(\nabla_\alpha X_\beta - \nabla_\beta X_\alpha)|_p$

$$\nabla_\alpha \nabla_\beta X_\gamma$$

$$= - \nabla_\alpha \nabla_\gamma X_\beta$$

$$= - \nabla_\gamma \nabla_\alpha X_\beta + R^\lambda{}_{\beta\alpha\gamma} X_\lambda$$

$$= \nabla_\gamma \nabla_\beta X_\alpha + \quad ''$$

$$= \nabla_\beta \nabla_\gamma X_\alpha + (- R^\lambda{}_{\alpha\gamma\beta} + R^\lambda{}_{\beta\alpha\gamma}) X_\lambda$$

$$= - \nabla_\beta \nabla_\alpha X_\gamma + ('')$$

$$= - \nabla_\alpha \nabla_\beta X_\gamma + (R^\lambda{}_{\gamma\beta\alpha} - R^\lambda{}_{\alpha\gamma\beta} + R^\lambda{}_{\beta\alpha\gamma}) X_\lambda$$

$$\overset{(*)}{=} - \nabla_\alpha \nabla_\beta X_\gamma - 2 R^\lambda{}_{\alpha\gamma\beta} X_\lambda$$

$$= - \nabla_\alpha \nabla_\beta X_\gamma + 2 R^\lambda{}_{\alpha\beta\gamma} X_\lambda$$

Where at $(*)$ we used the first Bianchi-Identity

$$R^\lambda{}_{\gamma\beta\alpha} + R^\lambda{}_{\beta\alpha\gamma} + R^\lambda{}_{\alpha\gamma\beta} = 0$$

This proves (3.45).

Since the $n + \frac{1}{2} n(n-1) = \frac{1}{2} n(n+1)$ values $X_\alpha(p)$, $(\nabla_\alpha X_\beta - \nabla_\beta X_\alpha)|_p$ determine all derivatives at p, the space of solutions of the linear equation (3.43) is at most $\frac{1}{2} n(n+1)$ dim. ■

Suppose $T_{\alpha\beta}$ satisfies Einstein Equation

$$G_{\alpha\beta} - \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}$$

so that $\quad T_{\alpha\beta} = T_{(\alpha\beta)} \qquad$ (3.46a)

and $\quad \nabla_\alpha T^{\alpha\beta} = 0; \qquad$ (3.46b)

and suppose further that K_α are the components of a Killing vector field so that

$$\nabla_{(\alpha} K_{\beta)} := \tfrac{1}{2}\left(\nabla_\alpha K_\beta + \nabla_\beta K_\alpha\right) = 0.$$

then

$$J^\alpha := T^{\alpha\beta} K_\beta \qquad (3.47)$$

are the components of a "conserved" (i.e. divergenceless) current:

$$\begin{aligned}
\nabla_\alpha J^\alpha &= \nabla_\alpha \left(T^{\alpha\beta} J_\beta\right) \\
&= \left(\nabla_\alpha T^{\alpha\beta}\right) J_\beta \\
&\quad + T^{\alpha\beta} \nabla_\alpha K_\beta \\
&= T^{\alpha\beta} \nabla_{(\alpha} K_{\beta)} \\
&= 0 \qquad (3.48)
\end{aligned}$$

Note

$$\nabla_\alpha J^\alpha = \partial_\alpha J^\alpha + \Gamma^\alpha_{\alpha\beta} J^\beta \qquad (3.49)$$

Now,

$$\Gamma^\alpha_{\alpha\beta} = \frac{1}{2} g^{\alpha\gamma} \left(- g_{\alpha\beta,\gamma} + g_{\gamma\alpha,\beta} + g_{\beta\gamma,\alpha} \right)$$

[by symmetry of $g^{\alpha\gamma} = g^{\gamma\alpha}$]

$$= \frac{1}{2} g^{\alpha\gamma} g_{\alpha\gamma,\beta}$$

$$= \frac{1}{2} \mathrm{Trace} \left(g^{-1} g_{,\beta} \right)$$

$$= \frac{1}{2} \frac{1}{\det(g)} \partial_\beta \det(g)$$

$$= \frac{1}{2} \partial_\beta \ln|\det(g)|$$

$$= \partial_\beta \ln \left(|\det(g)|^{1/2} \right)$$

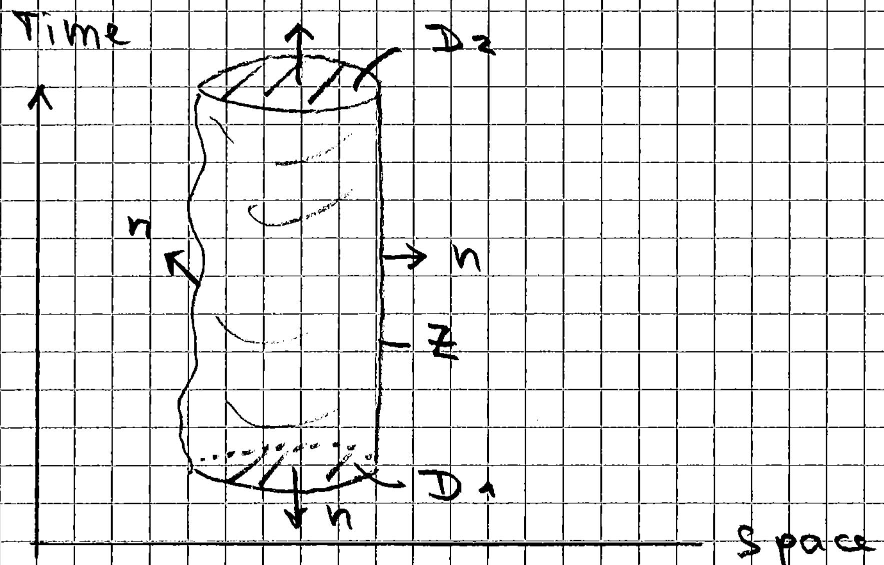
$$= \frac{1}{\sqrt{|\det(g)|}} \partial_\beta \sqrt{|\det(g)|}$$

$$\Rightarrow \nabla_\beta J^\beta = \frac{1}{\sqrt{g}} \partial_\beta \left(\sqrt{g} J^\beta \right) \qquad (3.50)$$

Stokes Theorem

$$0 = \int_{\Omega} \nabla_\beta J^\beta \sqrt{g}\, d^4x$$

$$= \int_{\partial\Omega} (J^\beta n_\beta) \sqrt{g}\, d^3x \qquad (3.51)$$



$$\partial\Omega = D_1 \cup D_2 \cup Z$$

$$= -(-D_1) \cup D_2 \cup Z \qquad (3.52)$$

$$\int_{D_2} (\ldots) - \int_{D_1} (\ldots) = - \int_{Z} (\ldots) \qquad (3.53)$$

$$(\ldots) = J^\beta n_\beta \sqrt{g}.$$

$\Leftrightarrow$ The excess of what goes out at D_2 to what comes in at D_1 must have come in through Z.

Stokes Theorem with forms

Let $\underline{J} := J_\alpha dx^\alpha$

be the current 1-form

$$* \underline{J} = J_\alpha * dx^\alpha =$$
$$= \frac{1}{3!} J^\alpha \varepsilon_{\alpha\beta\gamma\delta} dx^\beta \wedge dx^\gamma \wedge dx^\delta$$

$*$ = Hodge dual. Then

$$* d * \underline{J} = - \nabla_\alpha J^\alpha$$

and

$$0 = \int_\Omega d * \underline{J} = \int_{\partial\Omega} * \underline{J}$$

It is the 3-form $* \underline{J}$ that is to be integrated over the 3-dim boundary $\partial\Omega$.

If $T^{\alpha\beta}$ is the energy momentum tensor of a perfect fluid characterized by

$\rho : M \to \mathbb{R}_{\geq 0}$ (rest-mass density),

$p : M \to \mathbb{R}$ (rest-frame pressure),

$u^\alpha \in ST^1_0(M)$ (four velocity of fluid),

then

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + p \underbrace{\left(\frac{u^\alpha u^\beta}{c^2} - g^{\alpha\beta} \right)}_{h^{\alpha\beta}} \qquad (3.54a)$$

$$= \left(\rho + \frac{p}{c^2} \right) u^\alpha u^\beta - p g^{\alpha\beta} \qquad (3.54b)$$

Let $\{e_0, e_a\} = \{e_\alpha\}$ be o.n. basis, such that

$$\left. \begin{aligned} &g(e_\alpha, e_\beta) = \eta_{\alpha\beta} \\ &\text{with } e_0 = u/c \quad \text{(comoving)}, \end{aligned} \right\} \qquad (3.55)$$

$$\left. \begin{aligned} T_{00} &:= T(e_0, e_0) = c^2 \rho \\ T_{0a} &:= T(e_0, e_a) = 0 \\ T_{ab} &:= T(e_a, e_b) = p\, \delta_{ab} \end{aligned} \right\} \qquad (3.56)$$

$$\nabla_\alpha T^{\alpha\beta} = u^\beta \nabla_\alpha \left[\left(\rho + \frac{p}{c^2} \right) u^\alpha \right]$$

$$+ (\rho + p/c^2)\, u^\alpha \nabla_\alpha u^\beta$$

$$+ \nabla^\beta p \qquad (3.57)$$

Note $\quad u^\alpha \nabla_\alpha u^\beta = (\nabla_u u)^\beta = a^\beta \qquad (3.58)$

and $\quad g(u, a) = u_\alpha a^\beta = 0 \qquad (3.59)$

since $\quad g(u,u) = c^2 \quad \curvearrowright$

$\quad \nabla_u\, g(u,u) = 2\, g(u,a) = 0. \qquad$ } (3.60)

$$\nabla_\alpha T^{\alpha\beta} = \Big\{ u \left(\nabla \cdot \left(\rho + \frac{p}{c^2} \right) u \right)$$

$$+ a\, (\rho + p/c^2)$$

$$+ (\nabla p)^{\parallel} - (\nabla p)^{\perp} \Big\}^\beta \qquad (3.61)$$

where

$$\left[(\nabla p)^{\parallel} \right]^\beta = \frac{1}{c^2} u^\beta u^\alpha \nabla_\alpha p$$

$$=: \frac{1}{c^2} u^\beta \dot{p} \qquad (3.62)$$

$$\left[(\nabla p)^{\perp} \right]^\beta = \left(g^{\alpha\beta} - \frac{u^\alpha u^\beta}{c^2} \right) \nabla_\alpha p$$

$$= - h^{\alpha\beta} \nabla_\alpha p \qquad (3.63)$$

Hence $\nabla_\alpha T^{\alpha\beta} = 0$ is equivalent to

$$\nabla_\alpha \left(\left(\varrho + \frac{p}{c^2} \right) u^\alpha \right) - \frac{1}{c^2} \dot{p} = 0$$

or
$$\nabla_\alpha (\varrho u^\alpha) + \frac{p}{c^2} \nabla_\alpha u^\alpha = 0 \qquad (3.64a)$$

and

$$a^\alpha (\varrho + p/c^2) = - h^{\alpha\beta} \nabla_\beta p \qquad (3.64b)$$

The last equation is the relativistic <u>Euler equation</u>, generalising (1.21). Equation (3.64a) is almost a conservation for rest mass, which would be $\nabla_\alpha (\varrho u^\alpha) = 0$, but has an additional c^{-2}-term $\sim p \nabla_\alpha u^\alpha$ that accounts for the fact that a compression of the fluid under pressure p adds to its rest mass. (See GR-lecture SoSe 2020, Problem Sheet 3, Problem 1).