Lecture 4: Cosmological models in GR

In GR a cosmological model consists of the following data:

1.) A matter model $\rightarrow T_{\alpha\beta}$

2.) A solution $g_{\alpha\beta}(x^\lambda)$ to Einstein's Equation compatible with 1):

$$G_{\alpha\beta} - \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta} \qquad (4.1)$$

3.) A normalised timelike geodesic vector field (four-velocity) u

- $g_{\alpha\beta} u^\alpha u^\beta = c^2 \qquad (4.2)$
- $u^\alpha \nabla_\alpha u^\beta = a^\beta = 0 \qquad (4.3)$

Remark: The integral curves of u obey

$$\dot{x}^\alpha(\tau) = u^\alpha(x(\tau)) \qquad (4.4)$$

so that (4.3) says that $\tau \rightarrow x^\alpha(\tau)$ are geodesics:

$$\ddot{x}^\alpha(\tau) + \Gamma^\alpha_{\beta\gamma}(x(\tau))\,\dot{x}^\beta(\tau)\,\dot{x}^\gamma(\tau) = 0 \qquad (4.5)$$

and (4.2) says that

$$g_{\alpha\beta}(X(\tau))\,\dot{X}^{\alpha}(\tau)\,\dot{X}^{\beta}(\tau) = c^2 \qquad (4.6)$$

$$\Rightarrow \quad c\tau = s = \text{arc length}$$

$$\tau := s/c = \text{"Eigentime"} \qquad (4.7)$$

The intended meaning of 3.1 is that of the four-velocity field of "cosmological matter". Observers moving along the integral lines of u are "comoving observers". They are chosen to be geodesic because they should be freely falling without pressure forces. Then (3.46b) applies with $\nabla_\beta p = 0$. Note that for one and the same (M, g, T) we may chose different u and hence get different cosmological models. Sometimes this might give familiar spacetimes a strange look if we represent its metric in coordinates adapted to u. We will see examples of this later on and in the Exercise class ($\mapsto$ Milne universe).

The cosmological models we consider are restricted by the Copernican Principle, saying that every observer "sees" the same. In addition we want to restrict the class of copernican-principle-universes to those where every observer sees an <u>isotropic</u> universe.

How do we pose this condition mathematically?

Let

$$\mathrm{Isom}(M,g) := \{ \phi \in \mathrm{Diff}(M) : \phi^* g = g \} \quad (4.8)$$

and

$$\mathrm{Isom}_p(M,g) := \{ \phi \in \mathrm{Isom}(M,g) : \phi(p) = p \} \quad (4.9)$$

and

$$\mathrm{Isom}_{(p,u)}(M,g) := \{ \phi \in \mathrm{Isom}_p(M,g) : \phi_*|_p u = u(p) \} \quad (4.10)$$

Hence $\mathrm{Isom}_{(p,u)}(M,g)$ is the subgroup of isometries of (M,g) that fix the point $p \in M$ as well as the vector $u(p) \in T_pM$.

Note: If $\phi \in \mathrm{Isom}_p(M,g)$ then

$$\phi_*|_p : T_pM \to T_pM \qquad (4.11)$$

is a linear isometry of (T_pM, g_p). That is

$$\phi_*|_p \in O(T_pM, g_p) \qquad (4.12)$$

If $\phi \in \mathrm{Isom}_{(p,u)}$ then $\phi_*|_p$ fixes $u(p) \in T_pM$ and, since it acts by a g_p-preserving linear map, also its complement

$$[u(p)]^\perp := \{V \in T_pM : g(u,V)_p = 0\} \qquad (4.13)$$

Hence

$$\phi \in \mathrm{Isom}_{(p,u)}(M,g) \Rightarrow$$

$$\phi_*|_p \in O([u(p)]^\perp, g_p|_{u^\perp(p)}) \qquad (4.14)$$

Now, $[u(p)]^\perp \subset T_pM$ is the "rest space" of the co-moving observer (defined by u) at p. $O([u(p)]^\perp)$ is <u>this</u> observers orthogonal group. To say that this observer should experience "spatial isotropy" means that

$$\{\phi_*|_p : \phi \in \mathrm{Isom}_{(p,u)}(M,g)\} \supseteq SO([u(p)]^\perp, g_p|_{u^\perp(p)}). \quad (4.15)$$

The strong (i.e. including the isotropy requirement) Copernican principle then says that (4.15) holds for __all__ $p \in M$.

> **Strong Copernican principle:** The set of isometries of (M,g) comprises the set of rotations of each $[u(p)]^\perp \subset T_pM$ for all $p \in M$.

__Theorem__ If (M,g,u) is a cosmological model satisfying the strong (isotropic) Copernican Principle, then there exist local coordinates $(t,\vec{x})$ such that

$$u = \frac{\partial}{\partial t} \quad (4.16)$$

and

$$g = c^2\, dt \otimes dt - a^2(t)\, \frac{\sum_{a=1}^{3} dx^a \otimes dx^a}{\left(1 + \frac{k}{4} \rho^2\right)^2} \qquad (4.17)$$

Where

$$\rho^2 = \sum_{a=1}^{3} (x^a)^2 \qquad (4.18)$$

and $k \in \{1, 0, -1\}$ (4.19)

We could also say that

$$g = c^2\, dt \otimes dt - a^2(t)\, \hat{g} \qquad (4.20)$$

where $\hat{g}$ is a 3-dimensional Riemannian metric of constant (sectional) curvature

A proof of this may be found in

Norbert Straumann: "Minimal assumptions leading to a Robertson Walker model of the universe". Helvetica Physica Acta, year 1974, Volume 47, Issue 3, pages 379 - 383. See also the addendum to this lecture: pp. (4.25 - 4.39)

Alternative forms of the metric (4.17) are as follows

a) Set
$$X^1 = \varrho \sin\theta \cos\varphi$$
$$X^2 = \varrho \sin\theta \sin\varphi \qquad (4.21)$$
$$X^3 = \varrho \cos\theta$$

with
$$r := \frac{\varrho}{1+\frac{k}{4}\varrho^2} \qquad (4.22)$$

Then
$$\sum_{a=1}^{3} dX^a\, dX^a = d\varrho^2 + \varrho^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (4.23)$$

and
$$dr = \frac{d\varrho}{1+\frac{k}{4}\varrho^2} - \frac{\frac{k}{2}\varrho^2\, d\varrho}{(1+\frac{k}{4}\varrho^2)^2}$$

$$= \frac{(1-\frac{k}{4}\varrho^2)}{(1+\frac{k}{4}\varrho^2)^2}\, d\varrho \qquad (4.24)$$

$$d\varrho^2 = (1+\tfrac{k}{4}\varrho^2)^4 / (1-\tfrac{k}{4}\varrho^2)^2$$
$$\varrho^2 = (1+\tfrac{k}{4}\varrho^2)^2\, r^2$$

$$\Rightarrow \frac{\sum dX^a\, dX^a}{(1+\frac{k}{4}\varrho^2)^2} = \frac{(1+\frac{k}{4}\varrho^2)^2}{(1-\frac{k}{4}\varrho^2)^2}\, dr^2 + r^2 d^2\Omega \qquad (4.25)$$

Now

$$1 - k r^2 = 1 - k \frac{\varrho^2}{\left(1 + \frac{k}{4}\varrho^2\right)^2}$$

$$= \frac{\left(1 + \frac{k}{4}\varrho^2\right)^2 - k\varrho^2}{\left(1 + \frac{k}{4}\varrho^2\right)^2}$$

$$= \frac{1 - \frac{k}{4}\varrho^2}{\left(1 + \frac{k}{4}\varrho^2\right)^2} \qquad (4.26)$$

hence

$$\frac{\sum dx^a dx^a}{\left(1 + \frac{k}{4}\varrho^2\right)^2} = \frac{dr^2}{1 - k r^2} + r^2 d\Omega$$

$$\rightsquigarrow \boxed{g = c\,dt \otimes c\,dt - a^2(t)\left[\frac{dr^2}{1 - k r^2} + r^2 d^2\Omega\right]} \qquad (4.27)$$

Standard form of FLRW - Metric

Recall: $k \in \{1, 0, -1\}$

$k = +1$ "closed universe"
$k = 0$ "flat universe"
$k = -1$ "open universe"

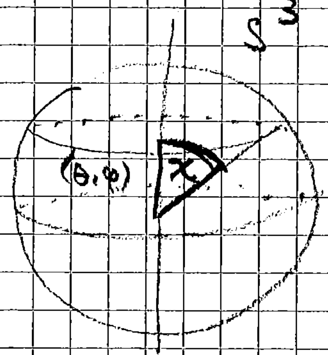
b) Set

$$r = \begin{cases} \sin\chi & \text{for } k = +1 \\ \chi & \text{for } k = 0 \\ \sinh(\chi) & \text{for } k = -1 \end{cases} \tag{4.28}$$

then

$$\frac{dr^2}{1 - kr^2} = d\chi^2 \tag{4.29}$$

so that



$$g = c^2 dt^2 - a^2(t)\left[d\chi^2 + \Sigma^2(\chi)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right] \tag{4.30}$$

with

$$\Sigma(\chi) := \begin{cases} \sin\chi & \text{for } k = +1 \\ \chi & \text{for } k = 0 \\ \sinh\chi & \text{for } k = -1 \end{cases} \tag{4.31}$$

Topology: Possibilities are

$$M^4 \cong \begin{cases} \mathbb{R} \times S^3 & \text{for } k = +1 \\ \mathbb{R} \times \mathbb{R}^3 & \text{for } k = 0 \text{ or } k = -1 \end{cases}$$

Geometry of FLRW metrics

<u>Definition</u>. An FLRW metric is of the form

$$g = c\,dt \otimes c\,dt - a^2(t)\,\hat{g} \qquad (4.32)$$

on the manifold

$$M = \mathbb{R} \times \hat{M} \qquad (4.33)$$

where $\hat{g}$ is a Riemannian metric on the 3-dimensional manifold $\hat{M}$ which is of constant sectional curvature $k = \{1, 0, -1\}$

$$\hat{R}_{abcd} = k\,(\hat{g}_{ac}\hat{g}_{bd} - \hat{g}_{ad}\hat{g}_{bc}) \qquad (4.34)$$

and where $a : \mathbb{R} \to \mathbb{R}_{>0}$ is a positive function.

In what follows, we shall determine the curvature tensor for (4.32) and, in particular, the Einstein tensor.

We write in terms of o.n. frame:

$$\hat{g} = \sum_{a=1}^{3} \hat{\theta}^a \otimes \hat{\theta}^a \qquad (4.35)$$

$$g = \theta^0 \otimes \theta^0 - \sum_{a=1}^{3} \theta^a \otimes \theta^a \qquad (4.36)$$

Where

$$\theta^0 = c\,dt \qquad (4.37a)$$

$$\theta^a = a(t)\,\hat{\theta}^a \qquad (4.37b)$$

We use the 1st and 2nd Cartan structure equation to compute the connection 1-forms from

$$\boxed{d\theta^\alpha + \omega^\alpha{}_\beta \wedge \theta^\beta = 0} \qquad (4.38)$$

(1st. Cartan Structure Eq.)

Where

$$\boxed{\omega_{\alpha\beta} = -\,\omega_{\beta\alpha}} \qquad (4.39)$$

with $\omega_{\alpha\beta} := \eta_{\alpha\lambda}\,\omega^\lambda{}_\beta$

and the curvature 2-forms from

$$\boxed{\Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\lambda \wedge \omega^\lambda{}_\beta} \qquad (4.40)$$

(2nd. Cartan Structure Eq.)

The components of the Riemann curvature tensor are obtained via the curvature 2-form via

$$\Omega^{\alpha}{}_{\beta} = \frac{1}{2} R^{\alpha}{}_{\beta\mu\nu}\, \theta^{\mu} \wedge \theta^{\nu} \qquad (4.41)$$

Recall that (4.38) expresses the <u>vanishing</u> <u>torsion</u> of $\omega^{\alpha}{}_{\beta}$ and (4.39) its <u>metricity</u>.

For the Riemannian (i.e. pos. definite) metric $\hat{g} = \Sigma_a \hat{\theta}^a \otimes \hat{\theta}^a$ we have the analogous equations

$$d\hat{\theta}^a + \hat{\omega}^a{}_b \wedge \hat{\theta}^b = 0 \qquad (4.42)$$

$$\hat{\omega}_{ab} = -\hat{\omega}_{ba} \qquad (4.43)$$

where $\hat{\omega}_{ab} := \hat{g}_{al}\, \hat{\omega}^l{}_b$ (4.44)

and $\hat{\Omega}^a{}_b = d\hat{\omega}^a{}_b + \hat{\omega}^a{}_l \wedge \hat{\omega}^l{}_b$ (4.45)

with $\hat{\Omega}^a{}_b = \hat{R}^a{}_{bmn}\, \hat{\theta}^m \wedge \hat{\theta}^n$ (4.46)

By hypothesis

$$R_{abcd} = \hat{g}_{al} R^l{}_{bcd} = k(\hat{g}_{ac}\hat{g}_{bd} - \hat{g}_{ad}\hat{g}_{bc}) \qquad (4.47)$$

($\Leftrightarrow$ $\hat{g}$ is of constant curvature)

Note that for the components of g and $\hat{g}$ with respect to orthonormal coframes $\{\theta^\alpha : \alpha = 0,1,2,3\}$ and $\{\hat{\theta}^a : a = 1,2,3\}$ we have

$$g_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1) \tag{4.48}$$

$$\hat{g}_{ab} = \mathrm{diag}(1,1,1). \tag{4.49}$$

This means that raising and lowering a spatial index with g results in a sign change, whereas raising and lowering with $\hat{g}$ gives no sign change. This has to be kept in mind later when, e.g., equations like

$$\omega^a{}_b = -\omega_{ab} \tag{4.50}$$

and

$$\hat{\omega}^a{}_b = +\hat{\omega}_{ab} \tag{4.51}$$

appear. Note

$$\omega_{ab} = g_{al}\,\omega^l{}_b = -\delta_{al}\,\omega^l{}_b, \tag{4.52}$$

$$\hat{\omega}_{ab} = \hat{g}_{al}\,\hat{\omega}^l{}_b = +\delta_{al}\,\hat{\omega}^l{}_b. \tag{4.53}$$

Now we calculate the connection 1-forms $\omega^\alpha{}_\beta$. From (4.37a) get

$$d\theta^0 = d(c\,dt) = 0$$
$$= -\omega^0{}_a \wedge \theta^a$$

$$\Rightarrow \quad \omega^0{}_a = \omega^a{}_0 \sim \theta^a \qquad (4.54)$$

Note that we have used

$$\omega^\alpha{}_\beta = 0 \text{ for } \alpha = \beta \qquad (4.55)$$

which follows from (4.39):

$$\omega^\alpha{}_\beta = \varepsilon_\alpha \, \omega_{\alpha\beta} = -\varepsilon_\alpha \omega_{\beta\alpha}$$
$$= -\varepsilon_\alpha \varepsilon_\beta \, \omega^\beta{}_\alpha \qquad (4.56)$$

where $\varepsilon_\alpha = g_{\alpha\alpha} \in \{\pm 1\}$. Hence for $\alpha = \beta \curvearrowright \varepsilon_\alpha \varepsilon_\beta = 1 \curvearrowright \omega^\alpha{}_\alpha = -\omega^\alpha{}_\alpha = 0$ (no summation over α).
We also used

$$\omega^0{}_a = \omega_{0a} = -\omega_{a0} = \omega^a{}_0 . \qquad (4.57)$$

Next from (4.37b) get

$$d\theta^a = \dot{a}\, dt \wedge \hat{\theta}^a + a\, d\hat{\theta}^a$$

$$= (\dot{a}/ca)\, \theta^0 \wedge \theta^a - a\, \hat{\omega}^a{}_b \wedge \hat{\theta}^b$$

↗ (4.42)

$$= -\left(\frac{\dot{a}}{ca}\right) \theta^a \wedge \theta^0 - \hat{\omega}^a{}_b \wedge \theta^b$$

$$= -\omega^a{}_\lambda \wedge \theta^\lambda$$

$$= -\omega^a{}_0 \wedge \theta^0 - \omega^a{}_b \wedge \theta^b \quad (4.58)$$

$$\Rightarrow \quad \omega^a{}_0 = \frac{\dot{a}}{ca}\, \theta^a + \sim \theta^0 \quad (4.59)$$

and $\omega^a{}_b = \hat{\omega}^a{}_b$ (4.60)

Now, (4.59) with (4.57) and (4.54) gives

$$\omega^a{}_0 = \omega^0{}_a = \left(\frac{\dot{a}}{ca}\right) \theta^a \quad (4.61a)$$

$$= (\dot{a}/c)\, \hat{\theta}^a \quad (4.61b)$$

This already determines all connection 1-forms as functions of a and the connection 1-forms of $\hat{g}$.

Before we proceed to calculate the curvature tensor we make a remark on the extrinsic curvature tensor of the spacelike hypersurface $\hat{M}$ of constant time t in M. If within $\hat{M} \subset M$ we take the tangent fields $e_a, e_b \in ST^1_0 \hat{M}$, normalized with respect to $g|_T$, we have

$$\nabla_{e_a} e_b = (\nabla_{e_a} e_b)^{\parallel} + (\nabla_{e_a} e_b)^{\perp} \qquad (4.62)$$

where $\parallel$ and $\perp$ refer to the components parallel and normal to $\hat{M}$ in M. Since $\hat{M}$ is of codimension 1, have

$$(\nabla_{e_a} e_b)^{\perp} = K(e_a, e_b)\, e_0 \qquad (4.63)$$

where K is a covariant symmetric tensor of rank 2 on $\hat{M}$, i.e. $K \in ST^0_2(\hat{M})$. It is called the extrinsic curvature of $\hat{M}$ in M.

From the general relation

$$\nabla_{e_a} e_b = [\omega(e_a)]^{\lambda}{}_{b}\, e_{\lambda}$$

$$= [\omega(e_a)]^{0}{}_{b}\, e_0 + \sim e_c \qquad (4.64)$$

We immediately read off

$$K(e_a, e_b) = [\omega(e_a)]^0{}_b$$
$$= i_{e_a} \omega^0{}_b$$
$$= i_{e_a}\left(\frac{\dot a}{ca}\theta^b\right)$$
$$= \left(\frac{\dot a}{ca}\right)\delta_{ab}$$
$$= -\left(\frac{\dot a}{ca}\right) g(e_a, e_b) \qquad (4.65)$$

Using (4.61). In invariant form this reads

$$K = -\left(\frac{\dot a}{ca}\right) g|_{T\hat{M}} \qquad (4.66a)$$
$$= +\left(\frac{a\dot a}{c}\right)\hat g \qquad (4.66b)$$

In any case, the extrinsic curvature is proportional to the metric, g or $\hat g$. Geometrically this means that the hypersurfaces of constant time t are <u>totally umbilic</u> (ext. curv. propto metric $\Leftrightarrow$ Weingarten map propto identity) and that the trace of K, the mean (extrinsic curvature) is proportional to Hubble parameter.

We now turn to the calculation of the Riemannian curvature

First the (0, a) – components:

$$\Omega^0{}_a = d\omega^0{}_a + \omega^0{}_n \wedge \omega^n{}_a \qquad (4.67)$$

1st. Term. Using (4.61b), we get

$$d\omega^0{}_a = d\left(\frac{\dot a}{c}\right) \wedge \hat\theta^a + \frac{\dot a}{c}\, d\hat\theta^a$$

$$= \frac{\ddot a}{c}\, dt \wedge \hat\theta^a - \frac{\dot a}{c}\, \hat\omega^a{}_{bc}\, \hat\theta^b \wedge \hat\theta^c$$

$$= \frac{\ddot a}{c^2 a}\, \theta^0 \wedge \theta^a - \frac{\dot a}{c a^2}\, \hat\omega^a{}_{bc}\, \theta^b \wedge \theta^c \qquad (4.68)$$

2nd Term. Aagain using (4.61b) and (4.60), we get

$$\omega^0{}_n \wedge \omega^n{}_a = \sum_n \frac{\dot a}{c}\, \hat\theta^n \wedge \hat\omega^n{}_{ba}\, \hat\theta^b$$

$$= -\frac{\dot a}{c}\, \hat\omega^a{}_{bn}\, \hat\theta^n \wedge \hat\theta^b$$

$$= \frac{\dot a}{c a^2}\, \hat\omega^a{}_{bc}\, \theta^b \wedge \theta^c \qquad (4.69)$$

Here we used $\hat\omega^n{}_{ba} = -\hat\omega^a{}_{bn}$ and we wrote $\sum_n$ explicitly in the first line to express summation over n which is not implied by summation convention since both indices are upstairs. Now, (4.68) and (4.69) together give in (4.67):

$$\Omega^0{}_a = \frac{\ddot{a}}{c^2 a}\, \theta^0 \wedge \theta^a \tag{4.70}$$

$$\Rightarrow R_{0a0b} = \left(\frac{\ddot{a}}{c^2 a}\right) \delta_{ab} \tag{4.71}$$

$$R_{0abc} = 0 \tag{4.72}$$

Next we turn to the spatial components.

$$\begin{aligned}\Omega^a{}_b &= d\omega^a{}_b + \omega^a{}_\lambda \wedge \omega^\lambda{}_b \\ &= d\hat{\omega}^a{}_b + \hat{\omega}^a{}_l \wedge \hat{\omega}^l{}_b \\ &\quad + \omega^a{}_0 \wedge \omega^0{}_b \\ &= \hat{\Omega}^a{}_b + \left(\frac{\dot{a}}{c\,a}\right)^2 \theta^a \wedge \theta^b \end{aligned} \tag{4.73}$$

using (4.45) and (4.61a).

Now, the constant curvature condition (4.34) gives

$$\hat{\Omega}^a{}_b = \frac{1}{2}\hat{R}^a{}_{bcd}\,\hat{\theta}^c \wedge \theta^d$$

$$= \frac{1}{2}k\left(\delta^a{}_c\,\hat{g}_{bd} - \delta^a{}_d\,\hat{g}_{bd}\right)\hat{\theta}^c \wedge \theta^d$$

$$= (k/2)\left(\hat{\theta}^a \wedge \hat{\theta}^b - \hat{\theta}^b \wedge \theta^a\right)$$

$$= (k/a^2)\;\theta^a \wedge \theta^b \qquad (4.74)$$

Since $\hat{g}_{bd} = \delta_{bd}$ etc.

Hence (4.73) gives

$$\Omega^a{}_b = a^{-2}\left(k + \frac{\dot{a}^2}{c^2}\right)\theta^a \wedge \theta^b$$

$$= \frac{1}{2}R^a{}_{b\mu\nu}\,\theta^\mu \wedge \theta^\nu$$

$$= -\frac{1}{2}R_{ab\mu\nu}\,\theta^\mu \wedge \theta^\nu \qquad (4.75)$$

leading to

$$R_{abcd} = -a^{-2}\left(k + \frac{\dot{a}^2}{c^2}\right)\left(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}\right) \qquad (4.76)$$

$$R_{abc0} = 0 \qquad (4.77)$$

Therefore, in total, the only non-vanishing components of the Riemann-tensor for the FLRW metric (4.32) are the following:

$$R_{0a0b} = \frac{\ddot{a}}{c^2 a}\,\delta_{ab} \qquad (4.78)$$

$$R_{abab} = -\left(k + \frac{\dot{a}^2}{c^2}\right)/a^2$$

(Curvature Comp. of FLRW)

The Kretschmann Scalar is as follows

$$K = R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu}$$

$$= 4 \cdot \sum_{a=1}^{3} R^{0a0a} R_{0a0a}$$

$$+ R^{abab} R_{abab}$$

$$= 4 \cdot 3 \, R^{0101} R_{0101}$$

$$+ 4 \cdot 3 \, R^{1212} R_{1212}$$

$$= \frac{12}{a^4}\left[\left(\frac{\ddot{a}a}{c^2}\right)^2 + \left(k + \frac{\dot{a}^2}{c^2}\right)^2\right] \qquad (4.79)$$

The Ricci-Tensor follows from

$$R_{\alpha\beta} = R^{\lambda}{}_{\alpha\lambda\beta}$$

$$= R_{0\alpha0\beta} - \sum_{l} R_{l\alpha l\beta} \tag{4.80}$$

which immediately gives

$$R_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta \tag{4.81}$$

and

$$R_{00} = -\sum_{l=1}^{3} R_{l0l0}$$

$$= -3\left(\ddot{a}/c^2 a\right) \tag{4.82}$$

$$R_{11} = R_{22} = R_{33}$$

$$= R_{0101} - R_{1212} - R_{1313}$$

$$= \frac{\ddot{a}}{c^2 a} + 2\,\frac{(\dot{a}^2/c^2) + k}{a^2} \tag{4.83}$$

For the Einstein tensor we have

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

hence immediately from (4.81):

$$G_{\alpha\beta} = 0 \quad \text{for } \alpha \neq \beta \tag{4.84}$$

The diagonal components follow as linear combinations of sectional curvatures:

$$\begin{aligned} G_{00} &= R_{00} - \tfrac{1}{2} g_{00} R \\ &= R_{00} - \tfrac{1}{2}(R_{00} - R_{11} - R_{22} - R_{33}) \\ &= \tfrac{1}{2}(R_{00} + 3 R_{11}) \\ &= \frac{3}{a^2}\left(k + \frac{\dot{a}^2}{c^2}\right) \end{aligned} \tag{4.85}$$

$$\begin{aligned} G_{11} &= R_{11} - \tfrac{1}{2} g_{11} R \\ &= R_{11} + \tfrac{1}{2}(R_{00} - R_{11} - R_{22} - R_{33}) \\ &= \tfrac{1}{2}(R_{00} + R_{11} - R_{22} - R_{33}) \\ &= \tfrac{1}{2}(R_{00} - R_{11}) \\ &= -2(\ddot{a}/c^2 a) - \frac{\dot{a}^2/c^2 + k}{a^2} \end{aligned} \tag{4.86}$$

and the same expressions for G_{22} and G_{33}. Together we have

$$G_{00} = \frac{3}{a^2}\left(k + \frac{\dot{a}^2}{c^2}\right)$$

$$G_{ab} = \left\{-2\,\frac{\ddot{a}}{c^2 a} - \frac{k + \dot{a}^2/c^2}{a^2}\right\}\delta_{ab} \qquad (4.87)$$

(Components of FLRW metric with respect to orthonormal basis)

In the following addendum to this Lecture 4 we shall prove that any metric of constant sectional curvature is of the form as the spatial part of the FLRW metric.

Addendum to Lecture 4

Metrics of constant curvature

Def. A metric g is said to be of constant curvature iff the sectional curvature

$$S(X,Y) := \frac{g(X, R(X,Y)Y)}{g(X,X)g(Y,Y) - [g(X,Y)]^2} \tag{4.88a}$$

$$= \frac{\mathrm{Riem}(X,Y,X,Y)}{g(X,X)g(Y,Y) - [g(X,Y)]^2} \tag{4.88b}$$

at each point p is independent of Span$\{X,Y\}$.

Note: $S(X,Y)$ depends on (X,Y) only via Span $\{X,Y\}$.

In components:

$$S(X,Y) = \frac{R_{\alpha\beta\mu\nu} X^\alpha X^\mu Y^\beta Y^\nu}{(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})X^\alpha X^\mu Y^\beta Y^\nu} \tag{4.88c}$$

The right-hand side is independent of (x,y) iff

$$R_{\alpha\beta\mu\nu} = k\,(g_{\alpha\mu}\, g_{\beta\nu} - g_{\alpha\nu}\, g_{\beta\mu}) \qquad (4.89)$$

for some function $k : M \to \mathbb{R}$.

The Ricci tensor and scalar are

$$R_{\alpha\beta} = k\,(n-1)\, g_{\alpha\beta} \qquad (4.90)$$

$$R = k\, n(n-1) \qquad (4.91)$$

and the Einstein tensor is:

$$G_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = \frac{k}{2}\,(2-n)(n-1)\, g_{\alpha\beta}. \qquad (4.92)$$

Proposition (Schur): If $n \geq 3$ then $k = \text{const}$.

Proof. From $\nabla^{\alpha} G_{\alpha\beta} \equiv 0$ and $\nabla_{\lambda} g_{\alpha\beta} = 0$ we immediately deduce from (4.92)

$$(2-n)(n-1)\,\nabla_{\beta} k = 0 \qquad (4.93)$$

$$\Leftrightarrow \quad k = \text{const} \quad \text{for} \quad n \geq 3. \qquad \blacksquare$$

Moreover, from the GR-Lecture 8 (SoSe 2020) we recall the definition of the Weyl-tensor, e.g. in the form of eq. (8.48) there:

$$\mathrm{Weyl} = \mathrm{Riem} - \frac{g \owedge \widetilde{\mathrm{Ric}}}{n-2} - R\,\frac{g \owedge g}{2n(n-1)} \qquad (4.94)$$

where $\widetilde{\mathrm{Ric}}$ is the trace-free part of Ric,

$$\widetilde{\mathrm{Ric}} = \mathrm{Ric} - \frac{1}{n}\, g\, R\,, \qquad (4.95)$$

and $\owedge$ denotes the Kulkarni-Nomizu product of two symmetric covariant rank 2 tensors, which in components reads (compare (8.37))

$$(h \owedge k)_{\alpha\beta\mu\nu} = h_{\alpha\mu} k_{\beta\nu} + h_{\beta\nu} k_{\alpha\mu} - h_{\alpha\nu} k_{\beta\mu} - h_{\beta\mu} k_{\alpha\nu} \qquad (4.96)$$

In particular

$$(g \owedge g)_{\alpha\beta\mu\nu} = 2\,(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}) \qquad (4.97)$$

In our case, (4.90) and (4.91) imply $\widetilde{\mathrm{Ric}} = 0$, whereas (4.89) and (4.91) imply

$$\mathrm{Riem} = R\,\frac{g \owedge g}{2n(n-1)} \qquad (4.97)$$

Hence (4.94) shows

$$\text{Weyl} = 0 \qquad (4.98)$$

Lemma: Constant curvature metrics have vanishing Weyl tensor; hence they are conformally flat*

$$g = \phi^2 \eta \qquad (4.99)$$

where η is the flat metric and $\phi : M \to \mathbb{R}_{>0}$ a nowhere vanishing (w.l.o.g. positive) function.

Now let x^α be local coordinates such that

$$\eta = \eta_{\alpha\beta}\, dx^\alpha \otimes dx^\beta \qquad (4.100a)$$

$$\eta = \text{diag}(\pm 1, \cdots, \pm 1) \qquad (4.100b)$$

Then

$$g = \eta_{\alpha\beta}\, \theta^\alpha \otimes \theta^\beta \qquad (4.101)$$

with $\theta^\alpha := \phi\, dx^\alpha$ (4.102)

The θ^α are the orthonormal coframe for g.

* Here we assume $n \geq 4$ since Weyl $\equiv 0$ for $n = 3$. But is remains true that const. curv $\Rightarrow$ conf. flat for $n = 3$.

Footnote *

In 3-dimensions the neccessary and sufficient condition for conformal flatness is not the vanishing of the Weyl-tensor, which vanishes identically in $n=3$ dimensions, but rather the vanishing of the Cotton-tensor

$$C_{\alpha\beta} = \varepsilon_\alpha{}^{\mu\nu} \nabla_\mu (R_{\nu\beta} - \tfrac{1}{4} R g_{\nu\beta})$$

But constant curvature metrics for $n=3$ have, according to (4.90-91)

$$R_{\alpha\beta} = 2k\, g_{\alpha\beta}$$
$$R = 6k$$

$$R_{\alpha\beta} - \tfrac{1}{4} R g_{\alpha\beta} = \frac{k}{2} g_{\alpha\beta}$$

with $k = $ const. (Schur). Hence $R_{\alpha\beta} - \frac{1}{4} R g_{\alpha\beta}$ is covariantly constant

$$\nabla_\mu (R_{\nu\beta} - \tfrac{1}{4} R g_{\nu\beta}) = 0$$

and in particular

$$C_{\alpha\beta} = 0$$

This proves conformal flatness in $n=3$ dimensions.

We use the Cartan structure equations to calculate the curvature tensor of g. We write $\phi_\alpha = \partial_\alpha \phi$, $\phi_{\alpha\beta} = \partial_\alpha \partial_\beta \phi$ etc.

$$d\theta^\alpha = \phi_\beta \, dx^\beta \wedge dx^\alpha$$
$$= -\frac{\phi_\beta}{\phi^2} \, \theta^\alpha \wedge \theta^\beta$$
$$= -\omega^\alpha{}_\beta \wedge \theta^\beta$$

$$\Rightarrow \omega^\alpha{}_\beta = \frac{\phi_\beta}{\phi^2} \theta^\alpha + \text{Terms} \sim \theta^\beta$$

$\omega_{\alpha\beta} = g_{\alpha\lambda} \omega^\lambda{}_\beta = -\omega_{\beta\alpha}$
then leads to

$$\omega^\alpha{}_\beta = \frac{\phi_\beta}{\phi^2} \theta^\alpha - \frac{g^{\alpha\lambda} \phi_\lambda}{\phi^2} g_{\beta\sigma} \theta^\sigma \qquad (4.103)$$

Using here that $\theta^\alpha = \phi \, dx^\alpha$ and that $g^{\alpha\lambda} g_{\beta\sigma} = \eta^{\alpha\lambda} \eta_{\beta\sigma}$ (since $g^{\alpha\lambda} = \phi^{-2} \eta^{\alpha\lambda}$ and $g_{\beta\sigma} = \phi^2 \eta_{\beta\sigma}$), we get

$$\omega^\alpha{}_\beta = \frac{\phi_\beta}{\phi} dx^\alpha - \frac{\phi^\alpha}{\phi} dx_\beta \qquad (4.104)$$

where $\phi^\alpha = \eta^{\alpha\beta} \phi_\beta$,
$dx_\beta = \eta_{\beta\alpha} dx^\alpha$

$$d\omega^\alpha{}_\beta = \left(\frac{\phi_\beta}{\phi}\right)_\gamma dx^\gamma \wedge dx^\alpha$$
$$- \left(\frac{\phi^\alpha}{\phi}\right)_\gamma dx^\gamma \wedge dx_\beta$$

$$= \left\{ \left(\frac{\phi_{\beta\gamma}}{\phi} - \frac{\phi_\beta \phi_\gamma}{\phi^2} \right) \delta^\alpha_\delta \right.$$
$$\left. - \left(\frac{\phi^\alpha{}_\gamma}{\phi} - \frac{\phi^\alpha \phi_\gamma}{\phi^2} \right) \eta_{\beta\delta} \right\} dx^\gamma \wedge dx^\delta \quad (4.105)$$

$$\omega^\alpha{}_\lambda \wedge \omega^\lambda{}_\beta = \left(\frac{\phi_\lambda}{\phi} dx^\alpha - \frac{\phi^\alpha}{\phi} dx_\lambda \right) \wedge$$
$$\left(\frac{\phi_\beta}{\phi} dx^\lambda - \frac{\phi^\lambda}{\phi} dx_\beta \right)$$

$$= \frac{\phi_\lambda \phi_\beta}{\phi^2} dx^\alpha \wedge dx^\lambda - \frac{\phi_\lambda \phi^\lambda}{\phi^2} dx^\alpha \wedge dx_\beta$$
$$- \frac{\phi^\alpha \phi_\beta}{\phi^2} \underbrace{dx_\lambda \wedge dx^\lambda}_{=0} + \frac{\phi^\alpha \phi^\lambda}{\phi^2} dx_\lambda \wedge dx_\beta$$

$$= \left\{ \frac{\phi_\beta \phi_\delta}{\phi^2} \delta^\alpha_\gamma + \frac{\phi^\alpha \phi_\gamma}{\phi^2} \eta_{\beta\delta} \right.$$
$$\left. - \frac{\phi_\lambda \phi^\lambda}{\phi^2} \delta^\alpha_\gamma \eta_{\beta\delta} \right\} dx^\gamma \wedge dx^\delta \quad (4.106)$$

Adding (4.105) and (4.106) and explicit antisymmetrisation in (γ, δ) then gives:

$$d\omega^\alpha{}_\beta + \omega^\alpha{}_\lambda \wedge \omega^\lambda{}_\beta =$$

$$\frac{1}{2}\Big\{ \underbrace{\frac{\phi_{\beta\gamma}}{\phi}\delta^\alpha{}_\delta}_{1} - \underbrace{\frac{\phi_{\beta\delta}}{\phi}\delta^\alpha{}_\gamma}_{2}$$

$$- \underbrace{\frac{\phi^\alpha{}_\gamma}{\phi}\eta_{\beta\delta}}_{3} + \underbrace{\frac{\phi^\alpha{}_\delta}{\phi}\eta_{\beta\gamma}}_{4}$$

$$- \underbrace{\frac{\phi_\beta\phi_\gamma}{\phi^2}\delta^\alpha{}_\delta}_{1} + \underbrace{\frac{\phi_\beta\phi_\delta}{\phi^2}\delta^\alpha{}_\gamma}_{2}$$

$$+ \underbrace{\frac{\phi^\alpha\phi_\gamma}{\phi^2}\eta_{\beta\delta}}_{3} - \underbrace{\frac{\phi^\alpha\phi_\delta}{\phi^2}\eta_{\beta\gamma}}_{4}$$

$$+ \underbrace{\frac{\phi_\beta\phi_\delta}{\phi^2}\delta^\alpha{}_\gamma}_{2} - \underbrace{\frac{\phi_\beta\phi_\gamma}{\phi^2}\delta^\alpha{}_\delta}_{1}$$

$$+ \underbrace{\frac{\phi^\alpha\phi_\gamma}{\phi^2}\eta_{\beta\delta}}_{3} - \underbrace{\frac{\phi^\alpha\phi_\delta}{\phi^2}\eta_{\beta\gamma}}_{4}$$

$$- \frac{\phi_\lambda\phi^\lambda}{\phi^2}\left(\delta^\alpha{}_\gamma\eta_{\beta\delta} - \delta^\alpha{}_\delta\eta_{\beta\gamma}\right)\Big\} * \qquad (4.107)$$

$$= \frac{1}{2}\Big\{ \left(\frac{\phi_{\beta\gamma}}{\phi} - 2\frac{\phi_\beta\phi_\gamma}{\phi^2}\right)\delta^\alpha{}_\delta$$

$$- \left(\frac{\phi_{\beta\delta}}{\phi} - 2\frac{\phi_\beta\phi_\delta}{\phi^2}\right)\delta^\alpha{}_\gamma$$

$$- \left(\frac{\phi^\alpha{}_\gamma}{\phi} - 2\frac{\phi^\alpha\phi_\gamma}{\phi^2}\right)\eta_{\beta\delta}$$

$$+ \left(\frac{\phi^\alpha{}_\delta}{\phi} - 2\frac{\phi^\alpha\phi_\delta}{\phi^2}\right)\eta_{\beta\gamma}$$

$$- \frac{\phi_\lambda\phi^\lambda}{\phi^2}\left(\delta^\alpha{}_\gamma\eta_{\beta\delta} - \delta^\alpha{}_\delta\eta_{\beta\gamma}\right)\Big\} * \qquad (4.108)$$

$$* = dx^\gamma \wedge dx^\delta$$

If we set

$$\psi = \frac{1}{\phi} \tag{4.109}$$

then

$$\psi_\alpha = -\frac{1}{\phi^2}\phi_\alpha$$

$$\psi_{\alpha\beta} = -\frac{1}{\phi^2}\phi_{\alpha\beta} + 2\frac{\phi_\alpha \phi_\beta}{\phi^3}$$

$$= -\psi\left(\frac{\phi_{\alpha\beta}}{\phi} - 2\frac{\phi_\alpha\phi_\beta}{\phi^2}\right) \tag{4.110}$$

hence

$$\frac{\phi_{\alpha\beta}}{\phi} - 2\frac{\phi_\alpha\phi_\beta}{\phi^2} = -\frac{\psi_{\alpha\beta}}{\psi} \tag{4.111}$$

and

$$\frac{\phi_\lambda \phi^\lambda}{\phi^2} = \frac{\psi^\lambda \psi_\lambda}{\psi^2} \tag{4.112}$$

Using that in (4.108) we get

$$\Omega^\alpha{}_\beta = \frac{1}{2}\left\{\delta^\alpha_\gamma \frac{\psi_{\beta\delta}}{\psi} - \delta^\alpha_\delta \frac{\psi_{\beta\gamma}}{\psi} + \eta_{\beta\delta}\frac{\psi^\alpha{}_\gamma}{\psi} - \eta_{\beta\gamma}\frac{\psi^\alpha{}_\delta}{\psi} - \frac{\psi_\lambda\psi^\lambda}{\psi^2}\left(\delta^\alpha_\gamma \eta_{\beta\delta} - \delta^\alpha_\delta \eta_{\beta\gamma}\right)\right\} dx^\gamma \wedge dx^\delta \tag{4.113}$$

Using the g-orthogonal basis

$$\theta^\alpha = \phi\, dx^\alpha = \psi^{-1} dx^\alpha \qquad (4.114)$$

$$\Omega^\alpha{}_\beta = \frac{1}{2}\Big\{ \psi\psi^\alpha{}_\gamma\, \eta_{\beta\delta} + \psi\psi_{\beta\delta}\, \delta^\alpha{}_\gamma - \psi\psi^\alpha{}_\delta\, \eta_{\beta\gamma} - \psi\psi_{\beta\gamma}\, \delta^\alpha{}_\delta - \psi_\lambda\psi^\lambda \left(\delta^\alpha{}_\gamma\, \eta_{\beta\delta} - \delta^\alpha{}_\delta\, \eta_{\beta\gamma}\right)\Big\}\, \theta^\gamma \wedge \theta^\delta$$

$$= \frac{1}{2} R^\alpha{}_{\beta\gamma\delta}\; \theta^\gamma \wedge \theta^\delta \qquad (4.115)$$

Hence the orthonormal components of the Riemann tensor are

$$R_{\alpha\beta\gamma\delta} = \psi\psi_{\alpha\gamma}\, \eta_{\beta\delta} + \psi\psi_{\beta\delta}\, \eta_{\alpha\gamma} - \psi\psi_{\alpha\delta}\, \eta_{\beta\gamma} - \psi\psi_{\beta\gamma}\, \eta_{\alpha\delta} - \psi_\lambda\psi^\lambda \left(\eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma}\right) \qquad (4.116)$$

or

$$\boxed{\mathrm{Riem} = \psi\, \partial^2\psi \circledwedge g - \frac{1}{2}\|d\psi\|_g^2\, g \circledwedge g} \qquad (4.117)$$

This is the general form of a conformally flat metrics $g = \varphi^{-2}\eta$ curvature tensor.

If this $R_{\alpha\beta\gamma\delta}$ is to be of constant curvature,

$$R_{\alpha\beta\gamma\delta} = k\,(g_{\alpha\gamma}\, g_{\beta\delta} - g_{\alpha\delta}\, g_{\beta\gamma}) \qquad (4.118)$$

or in orthonormal components $g_{\alpha\beta} = \eta_{\alpha\beta}$, we must have

$$\psi_{\alpha\gamma}\,\eta_{\beta\delta} + \psi_{\beta\delta}\,\eta_{\alpha\gamma}$$

$$- \psi_{\alpha\delta}\,\eta_{\beta\gamma} - \psi_{\beta\gamma}\,\eta_{\alpha\delta}$$

$$= \left[(k + \psi_{\lambda}\psi^{\lambda})/\psi\right](\eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma}) \qquad (4.119)$$

For $\alpha \neq \beta = \gamma \neq \delta \neq \alpha$ $(n \geq 3)$ the right-hand side is zero and on the left-hand side the only surviving term is $-\psi_{\alpha\delta}\,\eta_{\beta\gamma} = \pm\,\psi_{\alpha\delta}$.
Hence

$$\psi_{\alpha\gamma} = 0 \quad \text{for } \alpha \neq \gamma \qquad (4.120)$$

$$\Rightarrow \psi(x^1, \dots, x^n) = \sum_{\alpha=1}^{n} f_{\alpha}(x^{\alpha}) \qquad (4.121)$$

For $\alpha = \gamma \neq \beta = \delta$ (4.119) gives (no summation over repeated indices at same level):

$$\psi_{\alpha\alpha}\,\eta_{\beta\beta} + \psi_{\beta\beta}\,\eta_{\alpha\alpha} = \frac{k + \psi_{\lambda}\psi^{\lambda}}{\psi}\,\eta_{\alpha\alpha}\eta_{\beta\beta} \qquad (4.122)$$

Using (4.121) this becomes

$$f_\alpha'' \eta_{\beta\beta} + f_\beta'' \eta_{\alpha\alpha}$$

$$= \psi^{-1} \left[k + \sum_\lambda (f_\lambda')^2 \eta_{\lambda\lambda} \right] \eta_{\alpha\alpha} \eta_{\beta\beta} \qquad (4.123)$$

The left hand side depends only on x^α and x^β the right hand side on all $x^\gamma = (x^1, \dots, x^n)$. This is only possible if both sides are constant. Hence

$$\frac{k + \sum_\lambda (f_\lambda')^2 \eta_{\lambda\lambda}}{\psi} = c = \text{const.} \qquad (4.124)$$

$$\Leftrightarrow \quad k + \sum_\lambda (f_\lambda')^2 \eta_{\lambda\lambda} = c \sum_\lambda f_\lambda \qquad (4.125)$$

Differentiation of both sides wrt. x^α

$$\Rightarrow \quad \partial_\alpha (f_\alpha')^2 \eta_{\alpha\alpha} = c f_\alpha' \qquad (4.126)$$

$$\| \quad 2 f_\alpha' f_\alpha'' \eta_{\alpha\alpha} \qquad (4.127)$$

If $f_\alpha' \neq 0$ this implies $(\eta_{\alpha\alpha} = 1/\eta_{\alpha\alpha})$

$$f_\alpha'' = \frac{c}{2} \eta_{\alpha\alpha} \qquad (4.128)$$

$$\Rightarrow \quad f_\alpha(x^\alpha) = \frac{c}{4} (x^\alpha + a^\alpha)^2 \eta_{\alpha\alpha} + d^\alpha \qquad (4.129)$$

Choosing the origin of the x^α-coord. we may assume $a^\alpha = 0$. Setting

$$d := \sum_\alpha d^\alpha \qquad (4.130)$$

we set

$$\varphi(x) = \sum_\alpha \varphi_\alpha(x^\alpha) = \frac{c}{4} \sum_\alpha (x^\alpha)^2 \eta_{\alpha\alpha} + d$$

$$= \frac{c}{4} \eta_{\alpha\beta} x^\alpha x^\beta + d \qquad (4.131)$$

Equation (4.124), i.e.,

$$k + \varphi^\lambda \varphi_\lambda = c\varphi$$

then gives

$$k + \left(\frac{c}{2}\right)^2 x^\lambda \eta_{\lambda\alpha} x^\alpha = \frac{c}{4} \eta_{\alpha\beta} x^\alpha x^\beta + cd$$

$$\Leftrightarrow \quad k = cd \qquad (4.132)$$

and the metric

$$g = \varphi^{-2} \eta_{\alpha\beta}\, dx^\alpha \otimes dx^\beta \qquad (4.133)$$

reads

$$g = \frac{\eta_{\alpha\beta}\, dX^\alpha \otimes dX^\beta}{\left(d + \frac{c}{4}\eta_{\alpha\beta}X^\alpha X^\beta\right)^2} \tag{4.134}$$

Assuming $d \neq 0$ we can define

$$\bar{X}^\alpha := X^\alpha / d \tag{4.135}$$

and rewrite the metric, using $k = cd$, into

$$\boxed{g = \frac{\eta_{\alpha\beta}\, d\bar{X}^\alpha \otimes d\bar{X}^\beta}{\left(1 + \frac{k}{4}\eta_{\alpha\beta}X^\alpha X^\beta\right)^2}} \tag{4.136}$$

If $d = 0$ so that also $k = 0$, we have

$$g = \left(\frac{4}{c}\right)^2 \frac{\eta_{\alpha\beta}\, dX^\alpha \otimes dX^\beta}{[\eta(X,X)]^2} \tag{4.137}$$

Writing

$$X^\alpha := \frac{4}{c}\,\frac{\bar{X}^\alpha}{\eta(\bar{X},\bar{X})} \tag{4.138}$$

so that

$$\eta(X,X) = \left(\frac{4}{c}\right)^2 / \eta(\bar{X},\bar{X}) \tag{4.139}$$

i.e.

$$\eta(x,x)\,\eta(\bar{x},\bar{x}) = \left(\frac{4}{c}\right)^2. \qquad (4.140)$$

The inverse to (4.138) is

$$\bar{x}^\alpha = \frac{4}{c}\,\frac{x^\alpha}{\eta(x,x)} \qquad (4.141)$$

Then, taking the differential of (4.138) we get

$$dx^\alpha = \frac{4}{c}\,\frac{1}{\eta(\bar{x},\bar{x})}\left\{d\bar{x}^\alpha - 2\,\frac{\bar{x}^\alpha\,\bar{x}_\lambda\,d\bar{x}^\lambda}{\eta(\bar{x},\bar{x})}\right\} \qquad (4.142)$$

so that

$$\eta_{\alpha\beta}\,dx^\alpha \otimes dx^\beta = \left(\frac{4}{c}\right)^2 \frac{1}{[\eta(\bar{x},\bar{x})]^2}\,\eta_{\alpha\beta}\left(d\bar{x}^\alpha - 2\,\frac{\bar{x}^\alpha\,\bar{x}_\lambda\,d\bar{x}^\lambda}{\eta(\bar{x},\bar{x})}\right)\otimes\left(d\bar{x}^\beta - 2\,\frac{\bar{x}^\beta\,\bar{x}_\sigma\,d\bar{x}^\sigma}{\eta(\bar{x},\bar{x})}\right)$$

$$= \left(\frac{4}{c}\right)^2 \frac{\eta_{\alpha\beta}\,d\bar{x}^\alpha \otimes d\bar{x}^\beta}{[\eta(\bar{x},\bar{x})]^2} \qquad (4.143)$$

Hence

$$\frac{\eta_{\alpha\beta}\,dx^\alpha \otimes dx^\beta}{\left(\frac{c}{4}\right)^2 [\eta(x,x)]^2} = \left(\frac{4}{c}\right)^4 \frac{\eta_{\alpha\beta}\,d\bar{x}^\alpha \otimes d\bar{x}^\beta}{[\eta(x,x)\,\eta(\bar{x},\bar{x})]^2}$$

$$= \eta_{\alpha\beta}\,d\bar{x}^\alpha \otimes d\bar{x}^\beta \qquad (4.144)$$

using (4.140).

This shows that an inversion (4.138) transforms (4.134) for $d = 0$ into manifestly flat form. As a result we can state that (4.136) is valid for $k = 0$ as well.

Hence we have shown

Theorem: Let (M, g) be a $n \geq 3$ dimensional manifold with metric g of arbitrary signature and constant sectional curvature k. Then there exist local coordinates x^α s.t. g is of the form (4.136).

2. Addendum to Lecture 4

Conformal flatness of FLRW-Metrics

The FLRW-metric is

$$g = dx^0 \otimes dx^0 - a^2(x^0)\,\hat{g}$$

$$= a^2(\eta)\,[d\eta \otimes d\eta - \hat{g}] \quad (4.145)$$

where $d\eta = \frac{dx^0}{a(x^0)}$ (4.146)

$$= a^2(\eta)\,\tilde{g} \quad (4.147)$$

The Weyl-Tensor is conformally invariant, so $Weyl = 0 \iff \widetilde{Weyl} = 0$, where $\widetilde{Weyl}$ = Weyl-tensor of $\tilde{g}$.

The Riemann tensor for $\tilde{g}$ is easier to calculate that that for g and follows from the latter by setting $a(x^0) \equiv 1$ and identifying x^0 with η. Hence, using our calculations from Lecture 4, we get for the Riemann tensor of $\tilde{g}$:

From (4.72)

$$\tilde{R}_{0abc} = 0 \quad (4.148)$$

from (4.71)

$$\tilde{R}_{0a0b} = \left(\frac{\ddot{a}}{c^2 a}\right)\delta_{ab} \xrightarrow{a=1} 0 \quad (4.149)$$

from (4.76)

$$\tilde{R}_{abcd} = -\frac{k + \frac{\dot{a}^2}{c^2}}{a^2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})$$

$$\xrightarrow{a=1} = -k(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})$$

$$= -\hat{R}_{abcd} \quad (4.150)$$

Where

$$\hat{R}_{abcd} = k(\hat{g}_{ac}\hat{g}_{bd} - \hat{g}_{ad}\hat{g}_{bc}) \quad (4.151)$$

To sum up:

$$\tilde{R}^0{}_{abc} = 0 \quad (4.152)$$

$$\tilde{R}^0{}_{a0b} = 0 \quad (4.153)$$

$$\tilde{R}^a{}_{bcd} = \hat{R}^a{}_{bcd} \quad (4.154)$$

For the Ricci-tensor we get

$$\tilde{R}_{00} = \tilde{R}^{a}{}_{0ab} = 0 \quad (4.155)$$

$$\tilde{R}_{0a} = \tilde{R}^{\lambda}{}_{0\lambda a} = 0 \quad (4.156)$$

$$\tilde{R}_{ab} = \tilde{R}^{\lambda}{}_{a\lambda b} = \tilde{R}^{l}{}_{alb}$$

$$= \hat{R}^{l}{}_{alb} = \hat{R}_{ab} \quad (4.157)$$

And the Ricci scalar

$$\tilde{R} = \tilde{R}_{00} - \tilde{R}_{11} - \tilde{R}_{22} - \tilde{R}_{33}$$

$$= - \hat{R}_{11} - \hat{R}_{22} - \hat{R}_{33}$$

$$= - \hat{R} . \quad (4.158)$$

Now, the Weyl-tensor is given in terms of the Riemann- and Ricci-tensor and Ricci scalar in general n dimensions (GR-Lecture 8, 8.45)

$$W_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}$$

$$- \frac{1}{n-2}\left(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\alpha\delta}\right)$$

$$+ \frac{R}{(n-1)(n-2)}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right) \quad (4.159)$$

Applied to $\tilde{g}$ we get

$$\tilde{W}_{0a0b} = \tilde{R}_{0a0b} - \frac{1}{2}\left(\tilde{g}_{00}\tilde{R}_{ab} + \tilde{g}_{ab}\tilde{R}_{00} - \tilde{g}_{0b}\tilde{R}_{a0} - \tilde{g}_{a0}\tilde{R}_{0b}\right) + \frac{\tilde{R}}{6}\left(\tilde{g}_{00}\tilde{g}_{ab} - \tilde{g}_{0b}\tilde{g}_{a0}\right)$$

$$= -\frac{1}{2}\hat{R}_{ab} + \frac{(-\hat{R})}{6}(-\hat{g}_{ab})$$

$$= -\frac{1}{2}\left(\hat{R}_{ab} - \frac{1}{3}\hat{R}\right)$$

$$= 0 \qquad (4.160)$$

Since $\hat{g}$ is constant curvature and hence Einstein: $\hat{R}_{ab} = k\, g_{ab}$ $\Rightarrow \hat{R} = 3k \rightsquigarrow \hat{R}_{ab} - \frac{1}{3}\hat{R}\, g_{ab} = 0$.

Further we have

$$\tilde{W}_{0abc} = \tilde{R}_{0abc} - \frac{1}{2}\left(\tilde{g}_{0b}\tilde{R}_{ac} + \tilde{g}_{ac}\tilde{R}_{0b} - \tilde{g}_{0c}\tilde{R}_{ab} - \tilde{g}_{ab}\tilde{R}_{0c}\right) + \frac{\tilde{R}}{6}\left(\tilde{g}_{0b}\tilde{g}_{ac} - \tilde{g}_{0c}\tilde{g}_{ab}\right)$$

$$= 0 \quad (\tilde{g}_{0a} = \tilde{R}_{0a} = 0). \qquad (4.161)$$

Finally

$$\tilde{W}_{abcd} = \tilde{R}_{abcd}$$

$$- \frac{1}{2}\left(\tilde{g}_{ac} \tilde{R}_{bd} + \tilde{g}_{bd} \tilde{R}_{ac} - \tilde{g}_{ad} \tilde{R}_{bc} - \tilde{g}_{bc} \tilde{R}_{ad} \right)$$

$$+ \frac{\tilde{R}}{6}\left(\tilde{g}_{ac} \tilde{g}_{bd} - \tilde{g}_{ad} \tilde{g}_{bc} \right)$$

$$= - \hat{R}_{abcd}$$

$$+ \frac{1}{2}\left(\hat{g}_{ac} \hat{R}_{bd} + \hat{g}_{bd} \hat{R}_{ac} - \hat{g}_{ad} \hat{R}_{bc} - \hat{g}_{bc} \hat{R}_{ad} \right)$$

$$- \frac{\hat{R}}{6}\left(\hat{g}_{ac} \hat{g}_{bd} - \hat{g}_{ad} \hat{g}_{bc} \right)$$

$$= - \hat{W}_{abcd} \qquad (4.162)$$

But $\hat{g}$ being constant curvature has vanishing Weyl-tensor (see (4.98)).

Hence, we have shown

$$\tilde{W}_{\alpha\beta\gamma\delta} = 0 \qquad (4.163)$$

and hence

$$W_{\alpha\beta\gamma\delta} = 0 . \qquad (4.164)$$