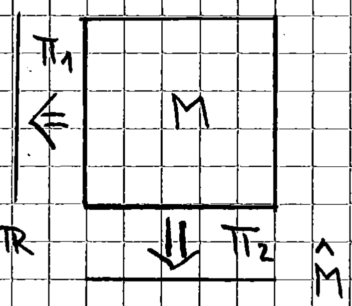
Lecture 5: Geodesics in FLRW-type Geometries

We consider space-time manifolds M which are topologically a product

$$M \cong \mathbb{R} \times \hat{M} \tag{5.1a}$$

where $\mathbb{R}$ represents "time" and $\hat{M}$ represents "space". The product structure means that there exist projection maps

$$\pi_1 : M \to \mathbb{R}$$
$$\pi_2 : M \to \hat{M} \tag{5.1b}$$



We give $\mathbb{R}$ the metric

$$g_{\mathbb{R}} = dx^0 \otimes dx^0 \tag{5.1c}$$

with respect to $x^0 : \mathbb{R} \to \mathbb{R}$ standard coordinate system and $\hat{M}$ a Riemannian metric

$$\hat{g} := g_{ab}\, dx^a \otimes dx^b . \tag{5.1d}$$

Let $a : \mathbb{R} \to \mathbb{R}_+$ (5.1e)

be a positive function depending only on "time".

Then we define the "warped product" metric ("warping function" is a) g on M by

$$g := \pi_1^* g_{\mathbb{R}} - (\pi_1^* a)^2 \, \pi_2^* \hat{g}$$
$$\text{"="} \; dx^0 \otimes dx^0 - a^2(x^0) \, \hat{g}_{ab} \, dx^a \otimes dx^b \qquad (5.1f)$$

Where "=" means that we now regard $\pi_1^* x^0 = x^0 \circ \pi_1$ and $\pi_2^* x^a = x^a \circ \pi_2$ as well as $\pi_1^* a = a \circ \pi_1$ and $\hat{g}_{ab} \circ \pi_2$ as functions on M denoted by x^0, x^a, a and $\hat{g}_{ab}$.

Metrics of the form (5.1f) generalise FLRW geometries insofar as $\hat{g}$ is not required to be of constant curvature

We now determine the Christoffel symbols for g:

$$\left.\begin{aligned} &\Gamma^{\alpha}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\lambda} \left(-g_{\beta\gamma,\lambda} + g_{\lambda\beta,\gamma} + g_{\gamma\lambda,\beta} \right) \\ &\text{Where} \quad g_{\alpha\beta,\lambda} := \partial g_{\alpha\beta} / \partial x^{\lambda} . \end{aligned}\right\} \quad (5.2)$$

Have

$$g_{00} = 1 = g^{00}, \qquad (5.3)$$

$$g_{0a} = 0 = g^{0a} \qquad (5.4)$$

Note that in (1+3) - block form:

$$g_{\alpha\beta} = \begin{pmatrix} 1 & \vec{0}^T \\ \vec{0} & -a^2 \hat{g}_{ab} \end{pmatrix} \qquad (5.5)$$

$$g^{\alpha\beta} = \begin{pmatrix} 1 & \vec{0}^T \\ \vec{0} & -a^{-2} \hat{g}^{ab} \end{pmatrix} \qquad (5.6)$$

Where

$$\hat{g}^{an} \hat{g}_{bn} = \delta^a_b \qquad (5.7)$$

Hence the components $\Gamma^\alpha_{\beta\gamma}$ are:

$$\Gamma^0_{00} = \frac{1}{2} g^{0\lambda} (-g_{00,\lambda} + g_{\lambda 0,0} + g_{0\lambda,0})$$

$$= 0 \qquad (5.8a)$$

$$\Gamma^a_{00} = \frac{1}{2} g^{a\lambda} (-g_{00,a} + g_{a0,0} + g_{0a,0})$$

$$= 0 \qquad (5.8b)$$

$$\Gamma^0_{0a} = \frac{1}{2} g^{0\lambda} (-g_{0a,\lambda} + g_{\lambda 0,a} + g_{a\lambda,0})$$

$$= \frac{1}{2} g^{0l} g_{al,0}$$

$$= 0$$

$$= \Gamma^0_{a0} \qquad (5.8c)$$

$$\Gamma^{a}_{00} = \frac{1}{2} g^{a\lambda} (-g_{00,\lambda} + g_{\lambda 0,0} + g_{0\lambda,0})$$

$$= 0 \qquad (5.8d)$$

$$\Gamma^{0}_{ab} = \frac{1}{2} g^{0\lambda} (-g_{ab,\lambda} + g_{\lambda a,b} + g_{b\lambda,a})$$

$$= \frac{1}{2} (-g_{ab,0} + g_{0a,b} + g_{b0,a})$$

$$= \frac{1}{2} (a^2 \hat{g}_{ab})_{,0}$$

$$= a a' \hat{g}_{ab} \qquad (5.8e)$$

$$\Gamma^{a}_{0b} = \frac{1}{2} g^{a\lambda} (-g_{0b,\lambda} + g_{\lambda 0,b} + g_{b\lambda,0})$$

$$= \frac{1}{2} g^{al} g_{bl,0}$$

$$= \frac{1}{2} (-a^{-2} \hat{g}^{al}) (-a^2 \hat{g}_{bl})_{,0}$$

$$= (a'/a) \delta^{a}{}_{b}$$

$$= \Gamma^{a}_{b0} \qquad (5.8f)$$

$$\Gamma^{a}_{bc} = \frac{1}{2} g^{a\lambda} (-g_{bc,\lambda} + g_{\lambda b,c} + g_{c\lambda,b})$$

$$= \frac{1}{2} g^{al} (-g_{bc,l} + g_{lb,c} + g_{cl,b})$$

$$= \frac{1}{2} (-a^{-2} \hat{g}^{al}) (-a^2) [-\hat{g}_{bc,l} + \hat{g}_{lb,c} + \hat{g}_{cl,b}]$$

$$= \frac{1}{2} \hat{g}^{al} (-\hat{g}_{bc,l} + \hat{g}_{lb,c} + \hat{g}_{cl,b})$$

$$= \hat{\Gamma}^{a}_{bc} \qquad (5.8g)$$

The geodesic equation for a curve

$$Z : \mathbb{R} \supseteq I \to M \tag{5.9}$$

in coordinate components

$$Z^\alpha := X^\alpha \circ Z : \mathbb{R} \supseteq I \to \mathbb{R} \tag{5.10}$$

is

$$\ddot{Z}^\alpha + (\Gamma^\alpha_{\beta\gamma} \circ Z)\, \dot{Z}^\beta \dot{Z}^\gamma = 0 \tag{5.11}$$

Here the composition of maps

$$\left.\begin{aligned} \Gamma^\alpha_{\beta\gamma} &: M \to \mathbb{R} \\ Z &: I \to M \\ \Rightarrow \quad \Gamma^\alpha_{\beta\gamma} \circ Z &: I \to \mathbb{R} \end{aligned}\right\} \tag{5.12}$$

means that we evaluate the $\Gamma^\alpha_{\beta\gamma}$ at the positions $Z(\lambda)$, $\lambda \in I \subseteq \mathbb{R}$.

For $\alpha = 0$ and $\alpha = a$, (5.11) reads in view of (5.8):

$$\ddot{Z}^0 + \underbrace{(\Gamma^0_{ab} \circ Z)}_{(5.8e)} \dot{Z}^a \dot{Z}^b = 0 \tag{5.13}$$

$$\ddot{Z}^0 + \left[(a\,a'\,\hat{g}_{ab})\circ Z\right]\dot{Z}^a\dot{Z}^b = 0 \qquad (5.14)$$

Note that a depends on X^0 only and that $a' = da/dX^0$. Hence, evaluated at $X = Z$ we have

$$a\,a'\big|_{X=Z} = a(Z^0)\,a'(Z^0) \qquad (5.15)$$

$$\hat{g}_{ab}\big|_{X=Z} = \hat{g}_{ab}(\vec{Z}) \qquad (5.16)$$

Since $\hat{g}_{ab}$ is independent of X^0.

For $\alpha = a$ we get for (5.11)

$$0 = \ddot{Z}^a + \underbrace{(\Gamma^a_{bc}\circ Z)}_{(5.8g)}\dot{Z}^c\dot{Z}^b + 2\underbrace{(\Gamma^a_{0b}\circ Z)}_{(5.8f)}\dot{Z}^0\dot{Z}^b$$

$$= \ddot{Z}^a + (\hat{\Gamma}^a_{bc}\circ Z)\dot{Z}^a\dot{Z}^b + 2\frac{a'}{a}\Big|_{X=Z}\dot{Z}^0\dot{Z}^a \qquad (5.17)$$

Hence we get for the spatial comp.

$$\ddot{Z}^a + (\hat{\Gamma}^a_{bc}\circ Z)\dot{Z}^a\dot{Z}^b = -2\frac{(a\circ Z)^{\dot{}}}{(a\circ Z)}\dot{Z}^a \qquad (5.18)$$

Here we wrote according to the chain rule: $(a\circ Z)^{\dot{}} = (a'\circ Z)\dot{Z}^0$. $\qquad (5.19)$

The geodesic equation implies

$$(g_{\alpha\beta} \circ Z)\, \dot{Z}^{\alpha} \dot{Z}^{\beta} = \varepsilon = \text{const} \qquad (5.20)$$

where

$$\varepsilon \begin{cases} > 0 & \text{timelike} \\ = 0 & \text{lightlike} \\ < 0 & \text{spacelike} \end{cases} \qquad (5.22)$$

Equations (5.14, 18, 20) contain all the information on geodesics in the geometry (5.1). Since the right-hand side of (5.18) is proportional to $\dot{Z}^a$ we see that the spatial projection of the geodesic into the hypersurface of constant X^0 is an autoparallel in that hypersurface, i.e. a geodesic that has been reparametrised into a <u>non-affine</u> form. We wish to investigate in more detail the relation between spacetime geodesics and their spatial projections. For this we need to investigate the behaviour of (5.14, 18, 20) under reparametrisations.

Reparametrisations

Let
$$f : \mathbb{R} \supseteq I \to I' \subseteq \mathbb{R} \tag{5.23}$$
be a C^2 - diffeomorphism; that is a change of parameter
$$I \ni \lambda \mapsto \sigma := f(\lambda) \in I' \tag{5.24}$$
The inverse to f is called
$$h : \mathbb{R} \supseteq I' \to I \subseteq \mathbb{R} \tag{5.25a}$$
$$h \circ f = id_I \tag{5.25b}$$
$$f \circ h = id_{I'} \tag{5.25c}$$

If $z : \mathbb{R} \supseteq I \to M$ is our original curve, it's reparametrisation via f is
$$\left.\begin{aligned} y &:= z \circ f^{-1} = z \circ h \\ &: \mathbb{R} \supseteq I' \to M \end{aligned}\right\} \tag{5.26}$$

Hence
$$z = y \circ f \tag{5.27a}$$
$$\dot{z} = (\dot{y} \circ f)\, \dot{f} \tag{5.27b}$$
$$\ddot{z} = (\ddot{y} \circ f)\, \dot{f}^2 + (\dot{y} \circ f)\, \ddot{f} \tag{5.27c}$$

Note that a "dot" denotes the derivative of the respective function with respect to its argument.

Inserting (5.27) into (5.14) gives

$$(\ddot{y}^a \circ f)\,\dot{f}^2 + (\dot{y}^a \circ f)\,\ddot{f}$$
$$+\left[(a\,a'\,\hat{g}_{ab}) \circ y \circ f\right](\dot{y}^a \circ f)(\dot{y}^b \circ f) = 0 \qquad (5.28)$$

or, equivalently

$$\ddot{y}^a + (a\,a'\,\hat{g}_{ab}) \circ y\;\dot{y}^a\,\dot{y}^b$$
$$= -\,\dot{y}^a\left(\frac{\ddot{f}}{\dot{f}^2} \circ h\right) \qquad (5.29)$$

where we composed the whole equation with $f^{-1} = h$ from the right. Now,

$$\frac{\ddot{f}}{\dot{f}^2} = -\left(\frac{1}{\dot{f}}\right)^{\cdot} \qquad (5.30)$$

and from (5.25b)

$$(h \circ f) = id|_I \;\rightsquigarrow\; (\dot{h} \circ f)\cdot \dot{f} = id \qquad (5.31)$$

Hence

$$\frac{1}{\dot f} = (\dot h \circ f)$$

$$-\left(\frac{1}{\dot f}\right)^{\cdot} = -(\ddot h \circ f)\,\dot f$$

$$= -\frac{\ddot h \circ f}{\dot h \circ f} = -\frac{\ddot h}{\dot h} \circ f \qquad (5.32)$$

therefore

$$\frac{\ddot f}{\dot f^{2}} \circ h = -\left(\frac{1}{\dot f}\right)^{\cdot} \circ h$$

$$= -\frac{\ddot h}{\dot h} \qquad (5.33)$$

and (5.29) becomes

$$\ddot y^{o} + \left[(a\, a'\, \hat g_{ab}) \circ y\right] \dot y^{a}\, \dot y^{b}$$

$$= \dot y^{o}\; \ddot h / \dot h \qquad (5.34)$$

In the same fashion (5.18) is re-written using (5.27):

$$(\ddot{y}^a \circ f)\,\dot{f}^2 + (\dot{y}^a \circ f)\,\ddot{f}$$

$$+ (\hat{\Gamma}^a_{bc} \circ y \circ f)(\dot{y}^a \circ f)(\dot{y}^b \circ f)\,\dot{f}^2$$

$$= -2\,\frac{(a \circ y \circ f)^{\circ}}{a \circ y \circ f}\,(\dot{y}^a \circ f)\,\dot{f}$$

$$= -2\left[\frac{(a \circ y)^{\cdot}}{(a \circ y)}\,\dot{y}^a\right] \circ f\ \dot{f}^2 \qquad (5.35)$$

where we used the chain-rule on the right-hand side:

$$(a \circ y \circ f)^{\circ} = ((a \circ y)^{\circ} \circ f)\,\dot{f}\,.$$

Again dividing (5.35) by $\dot{f}^2$ and composing the whole equation with $f^{-1} = h$ from the right, we get

$$\ddot{y}^a + (\hat{\Gamma}^a_{bc} \circ y)\,\dot{y}^a\,\dot{y}^b$$

$$= \left[-2\,\frac{(a \circ y)^{\cdot}}{(a \circ y)} - \frac{\ddot{f}}{\dot{f}^2} \circ h\right]\dot{y}^a$$

$$= \left[\frac{\ddot{h}}{\dot{h}} - 2\,\frac{(a \circ y)^{\cdot}}{(a \circ y)}\right] \qquad (5.36)$$

Using (5.33) on the right-hand side

Finally, (5.20) rewritten in terms of y reads

$$(g_{\alpha\beta} \circ y \circ f)(\dot{y}^\alpha \circ f)(\dot{y}^\beta \circ f)\, \dot{f}^2 = \varepsilon \qquad (5.37)$$

Again dividing by $\dot{f}^2$ and composing with $f^{-1} = h$ from right gives, using

$$\left(\frac{1}{\dot{f}^2}\right) \circ h = \dot{h}^2 \qquad (5.38)$$

which follows from (5.31),

$$(g_{\alpha\beta} \circ y)\, \dot{y}^\alpha \dot{y}^\beta = \dot{h}^2\, \varepsilon . \qquad (5.39)$$

We have now rewritten (5.14, 18, 20) for z in terms of $y = z \circ h$ resulting in (5.34, 36, 39) respectively.

From (5.36) it is immediately apparent how the new parameter $\sigma := f(\lambda)$ should be chosen such that $\sigma \mapsto y^a(\sigma) = (z^a \circ h)(\sigma)$ is a geodesic on 3-dim "space" $\hat{M}$ in the Riemannian metric $\hat{g}$ (attention: $\hat{g}$, not $a^2 \hat{g}$ = spatial part of g).

For a given function $a: M \to \mathbb{R}_+$ we can make the right-hand side of (5.36) to vanish – and hence the spatial projection $\Pi_2 \circ y : \sigma \mapsto y^a(\sigma) = z^a(\lambda(\sigma)) = z^a \circ h(\sigma)$ a geodesic of $(\hat{M}, \hat{g})$ – iff

$$\frac{\ddot{h}}{\dot{h}} = 2 \frac{(a \circ y)^\circ}{a \circ y} \qquad (5.40)$$

$$\Leftrightarrow \quad \left[\ln(\dot{h}) - 2 \ln(a \circ y)\right]^\circ = 0$$

$$\Leftrightarrow \quad \dot{h} = k\,(a \circ y)^2, \quad k = \text{const.} \qquad (5.41)$$

Since $\dot{h} = d\lambda / d\sigma$ this will give us the space-time affine parameter λ as function of the space affine parameter σ if we know $(a \circ y)$ as function of σ, i.e. for known a if we also knew $y^0(\sigma)$. But $y^0(\sigma) = z^0(\lambda(\sigma))$, so from $z^0(\lambda)$ we cannot deduce $y^0(\sigma)$ unless we know $\lambda(\sigma)$. Hence, for known $z^0(\lambda)$, we proceed differently by rewriting (5.41) in an equivalent form for $f = h^{-1}$ rather than h.

First we compose (5.41) with $h^{-1} = f$. Then

$$\dot{h} \circ f = k\,(a \circ y \circ f)^2$$

$$= k\,(a \circ z)^2 \qquad (5.42)$$

But from

$$h \circ f = id_I$$

get

$$(\dot{h} \circ f) = \frac{1}{\dot{f}} \qquad (5.43)$$

and (5.41) is equivalent to

$$\frac{1}{\dot{f}} = \frac{1}{\frac{d\sigma}{d\lambda}} = \frac{d\lambda}{d\sigma} = k\, a^2(z^a(\lambda)) \qquad (5.44)$$

$$\sigma(\lambda) = \frac{1}{k} \int_{\lambda_0}^{\lambda} \frac{d\lambda}{a^2(z^a(\lambda))} \qquad (5.45)$$

This determines the parameter σ as function of the old parameter λ so that $y^a(\sigma) = z^a(\lambda(\sigma))$ is a geodesic of $(\hat{M}, \hat{g})$ if $z^a(\lambda)$ is a geodesic of (M, g).

Our defining property of the parameter $\sigma = f(\lambda)$ was that it made the reparametrised spatial projection

$$\pi_2 \circ z \circ f^{-1} = \pi_2 \circ y \tag{5.46}$$

which we wrote as $y^a(\sigma)$, a geodesic of $(\hat{M}, \hat{g})$. Hence the parameter σ must be affinely equivalent to the Riemannian arc length of $(\hat{M}, \hat{g})$. Let us without loss of generality choose it to be equal to arc length. Then

$$(\hat{g}_{ab} \circ y)\, \dot{y}^a \dot{y}^b = 1 \tag{5.47}$$

Now, (5.39) reads

$$(\dot{y}^0)^2 - (a \circ y)^2 (\hat{g}_{ab} \circ y)\, \dot{y}^a \dot{y}^b = \dot{h}^2\, \varepsilon \tag{5.48}$$

Inserting (5.47) gives

$$(\dot{y}^0)^2 = (a \circ y)^2 + \dot{h}^2\, \varepsilon$$

$$= (a \circ y)^2 \left[1 + \varepsilon k^2 (a \circ y)^2 \right] \tag{5.49}$$

where we used (5.41) to eliminate $\dot{h}$ on the right-hand side. If we assume

$\dot{y}^0 > 0$, which for time- and lightlike geodesics just restricts to future pointing geodesics, we get

$$\frac{dy^0}{d\sigma} = a(y^0(\sigma)) \left[1 + \varepsilon k^2 a^2(y^0(\sigma))\right]^{1/2} \qquad (5.50)$$

This equation allows to determine σ as function of y^0, and hence y^0 as function of σ, via seperation of variables and integration

$$\sigma(y^0) = \int_c^{y^0} \frac{dx}{a(x)\left[1 + \varepsilon k^2 a^2(x)\right]^{1/2}} \qquad (5.51)$$

The integration of the geodesic problem for (M, g) can now be tackeled in a 2-step process:

1. Step: Integrate the Riemannian geodesic problem for $(\hat{M}, \hat{g})$ yielding $y^a(\sigma)$, σ = proper length of $\hat{g}$;

2. Step: Integrate (5.51) to obtain $y^0(\sigma)$ for timelike ($\varepsilon = +1$) or lightlike ($\varepsilon = 0$) geodesics in (M, g) (for which $\dot{y}^0 \neq 0$).

Once we have $y^0(\sigma)$ we can use it in (5.41) to determine the space-time affine parameter λ in terms of σ:

$$\dot{h} = \frac{d\lambda}{d\sigma} = k\, a^2(y^0(\sigma))$$

i.e. $$\lambda = \int^{\sigma} k\, d\sigma'\, a^2(y^0(\sigma'))$$

Most of this also applies to the physically less interesting case of spacelike geodesics in (M, g), as long as $\dot{y}^0 > 0$. Geodesics for $\dot{y}^0(\sigma) = 0$ i.e. $y^0(\sigma) = \text{const.}$ only exist if the $t = \text{const.}$ hypersurface is totally geodesic, which means that its extrinsic curvature K_{ab} must be zero. From (4.66) we know that this is true if $a'(x^0) = 0$ (denoted by $\dot{a}$ in Lecture 4).

This we see also from (5.34): If $y^0(\sigma) = \text{const.}$, then $\dot{y} = \ddot{y} = 0$ which implies $a'(y^0(\sigma)) = 0$, i.e. the hypersurface $X^0 = c = \text{const.}$ in which $y^\alpha(\sigma)$ lies must be such that $a'(X^0 = c) = 0$.

Our formulae are particularly simple for lightlike geodesics in (M,g), i.e. for $\varepsilon = 0$. Then the second step merely amounts to integrating

$$\sigma(y^0) = \int^{y^0} \frac{dx}{a(x)} =: \eta(y^0)$$

The coordinate η defined by

$$d\eta = \frac{dx^0}{a(x^0)}$$

is called "conformal time" because

$$\hat{g} = dx^0 \otimes dx^0 - a^2(x^0)\,\hat{g}$$

$$= a^2(x^0)\left[d\eta \otimes d\eta - \hat{g}\right]$$

and for any lightlike curve

$$\lambda \mapsto (\eta(\lambda), x^a(\lambda))$$

have

$$\dot{\eta}^2 = \hat{g}_{ab}\,\dot{x}^a \dot{x}^b = \left(\frac{d\sigma}{d\lambda}\right)^2$$

i.e. $d\eta = \pm\, d\sigma$

conf. time

spatial proper length