Lecture 6: The Friedmann Equations in GR

We wish to solve

$$G_{\alpha\beta} - g_{\alpha\beta}\Lambda = \kappa T_{\alpha\beta} \qquad (6.1)$$

where $\kappa = 8\pi G / c^4$ (6.2)

and
$$T^{\alpha\beta} = \rho u^\alpha u^\beta + \left(\frac{u^\alpha u^\beta}{c^2} - g^{\alpha\beta}\right) p$$
$$= (\rho + p/c^2) u^\alpha u^\beta - g^{\alpha\beta} p \qquad (6.3)$$

for
$$g_{\alpha\beta} = dx^0 \otimes dx^0 - a^2(x^0)\,\hat{g}$$
with $\hat{g} = \hat{g}(\vec{x})\, dx^a \otimes dx^b$ (6.4)

is of constant sectional curvature $k \in \{1, -1, 0\}$.

Now, let $\{e_0, e_1, e_2, e_3\}$ be any orthonormal basis

$$g_{\alpha\beta} = g(e_\alpha, e_\beta) = \eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1) \qquad (6.5)$$

with $e_0 = u/c$ (6.6)

then from (4.87)

$$G_{00} = \frac{3}{a^2}\left(k + \frac{\dot{a}^2}{c^2}\right) \qquad (6.7)$$

$$G_{ab} = \left\{ -2\frac{\ddot{a}}{c^2 a} - \frac{k + \dot{a}^2/c^2}{a^2} \right\} \delta_{ab} \qquad (6.8)$$

and

$$T_{00} = \rho c^2 \qquad (6.9)$$

$$T_{ab} = p\, \delta_{ab} \qquad (6.10)$$

Hence Einstein's equations are equivalent to

$$\frac{3}{a^2}\left(k + \frac{\dot{a}^2}{c^2}\right) - \Lambda = \frac{8\pi G}{c^4} c^2 \rho$$

$$\Leftrightarrow \quad \dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k c^2 + \frac{\Lambda c^2}{3} a^2 \qquad (6.11)$$

$$-2\frac{\ddot{a}}{c^2 a} - \frac{k + \dot{a}^2/c^2}{a^2} + \Lambda = \frac{8\pi G}{c^4} p$$

$$\Leftrightarrow \quad \ddot{a} = -\frac{\dot{a}^2}{2a} - \frac{4\pi G}{c^2} a p - \frac{k c^2}{2a} + \frac{\Lambda c^2}{2} a \qquad (6.12)$$

If we eliminate $\dot{a}^2$ in (6.12) via (6.11) we get

$$\ddot{a} = -\frac{1}{2a}\left(\frac{8\pi G}{3}\rho a^2 - kc^2 + \frac{\Lambda c^2}{3}a^2\right) - \frac{4\pi G}{c^2} a p - \frac{kc^2}{2a} + \frac{\Lambda c^2}{2} a$$

$$= -\frac{4\pi G}{3} a \left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} a \qquad (6.13)$$

Equations (6.11) and (6.13) together are equivalent to Einstein's equations and are called the Friedmann equations.

A useful consequence of (6.11) and (6.13) comes about as follows: Thake the time derivative of (6.11):

$$2\dot{a}\ddot{a} = 2\frac{8\pi G}{3}\rho a\dot{a} + 2\frac{\Lambda c^2}{3} a\dot{a} + \frac{8\pi G}{3} a^2 \dot{\rho} \qquad (6.14)$$

Divide by $2\dot{a}$ and replace $\ddot{a}$ by (6.13) on the left-hand side:

$$-\frac{4\pi G}{3} a\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} a$$

$$= \frac{8\pi G}{3}\rho a + \frac{\Lambda c^2}{3} a + \frac{4\pi G}{3} a^2 \frac{\dot{\rho}}{\dot{a}}$$

$$\Leftrightarrow -4\pi G\, a\left(\rho + \frac{p}{c^2}\right) = \frac{4\pi G}{3} a^2 \frac{\dot{\rho}}{\dot{a}}$$

$$\Leftrightarrow a\dot{\rho} + 3\dot{a}\rho + 3\dot{a}\frac{p}{c^2} = 0$$

$$\Leftrightarrow (a^3\rho)^{\cdot} + (a^3)^{\cdot}\frac{p}{c^2} = 0 \qquad 6.15$$

Hence the total system obtained is :

$$\boxed{\begin{aligned} \ddot{a} &= -\frac{4\pi G}{3} a\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} a && (6.16a)\\ \dot{a}^2 &= \frac{8\pi G}{3}\rho a^2 - kc^2 + \frac{\Lambda c^2}{3} a^2 && (6.16b)\\ (a^3\rho)^{\cdot} &+ (a^3)^{\cdot}\frac{p}{c^2} = 0 && (6.16c)\end{aligned}}$$

(The Friedmann Equations)

If in (6.16) we neglect p/c^2 against ϱ

$$|p/c^2| \ll \varrho \tag{6.17}$$

then (6.16 c) reads $(a^3 \varrho)^{\cdot} = 0$ and hence

$$\frac{4\pi}{3} \varrho a^3 = M = \text{const.} \tag{6.18}$$

Using this to replace ϱ in (6.16 a) and (6.16 b), where in (6.16 a) we again neglect $3p/c^2$ against ϱ, we obtain

$$\ddot{a} = -\frac{GM}{a^2} + \frac{\Lambda c^2}{3} a \tag{6.19}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{2GM}{a^3} - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \tag{6.20}$$

which exactly correspond to the Newtonian Friedmann equations (2.20) and (2.22) of Lecture 2, repectively, if we identify R there with a here and rename Λ and k there into Λc^2 and $k c^2$ here. Note that here Λ has the physical dimension of $[\text{length}]^{-2}$, i.e. curvature and

$k \in \{1, -1, 0\}$ is dimensionless (unit curvature). In the Newtonian case Λ has dimension [duration]$^{-2}$ and k [velocity]^2. In both cases the scale factor, R there and a here, has dimension [length]. Note that means that $\hat{g}$ is dimensionless so that $a^2 \hat{g}$ has dimension [length]^2. Therefore the spatial coordinates X^a are dimensionless (and hence $\hat{g}$ has a dimensionless curvature k) whereas X^0 has dimension [length].

As we have seen (6.16 a-c) are not independent. We have seen

(6.16 a) + (6.16 b) $\Rightarrow$ (6.16 c)

It is also immediate that

(6.16 b) + (6.16 c) $\Rightarrow$ (6.16 a)

This means that (6.16) constitute 2 independent ordinary differential equations of first order for the 3 functions ρ, p and a of t.

Hence the system (6.16) is <u>under-determined</u>. What is missing is an <u>equation of state</u> for the matter, which usually is of the form

$$p = p(\rho) \tag{6.22}$$

In cosmology two cases are of particular interest

1.) $p = 0$ "Dust"

$$(6.16c) \Rightarrow \quad \rho a^3 = \text{const} \tag{6.23}$$

2.) $p = \frac{1}{3}\rho c^2$ "Radiation"

$$(6.16c) \Rightarrow \quad \rho a^4 = \text{const} \tag{6.24}$$

In general one considers mixtures of matter

$$\rho = \sum_i \rho_i \, , \quad p = \sum_i p_i \tag{6.25}$$

the components of which satisfy

$$p_i = w_i \rho_i c^2 \tag{6.26}$$

for w_i some numbers.

The velocity of sound in a medium satisfying (6.26) satisfies

$$V_i^2 = \left| \frac{dp_i}{d\rho_i} \right| = |W_i|\, c^2 \qquad (6.27)$$

If we want $V_i^2 \leq c^2$ we must have

$$-1 \leq W_i \leq 1 \qquad (6.28)$$

From (6.16c) we get for a component i satisfying (6.26):

$$(a^3 \rho_i)^{\bullet} + (a^3)^{\bullet}\, W_i\, \rho_i = 0$$

$$\Leftrightarrow \quad (a^3)^{\bullet} (1 + W_i)\, \rho_i + a^3 \dot{\rho}_i = 0$$

$$\Leftrightarrow \quad \frac{\dot{\rho}_i}{\rho_i} = -3\,(1 + W_i)\, \frac{\dot{a}}{a}$$

$$\Leftrightarrow \quad \rho_i\, a^{3(1+W_i)} = \text{const.} \qquad (6.29)$$

Note that (6.28) implies that $(1+W_i) \geq 0$ so that ρ_i never increases with increasing a. The limiting case is $W_i = -1$, in which case $\rho_i =$ const, indep. of t. This corresponds to "matter" as represented by a cosm. constant.

In fact, writing

$$G_{\alpha\beta} - \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}$$

$$\Leftrightarrow \quad G_{\alpha\beta} = \kappa \left(T_{\alpha\beta} + \frac{\Lambda}{\kappa} g_{\alpha\beta} \right)$$

$$= \kappa \left(T_{\alpha\beta} + T^{\Lambda}_{\alpha\beta} \right) \qquad (6.30)$$

with $$T^{\Lambda}_{\alpha\beta} = \frac{\Lambda}{\kappa} g_{\alpha\beta} \qquad (6.31)$$

This is indeed of perfect-fluid form

$$T^{\Lambda}_{\alpha\beta} = u_\alpha u_\beta \, \rho_\Lambda + \left(\frac{u_\alpha u_\beta}{c^2} - g_{\alpha\beta} \right) p_\Lambda$$

$$= \left(\rho_\Lambda + p_\Lambda / c^2 \right) u_\alpha u_\beta - g_{\alpha\beta} \, p_\Lambda$$

$$\stackrel{!}{=} \frac{\Lambda}{\kappa} g_{\alpha\beta} \qquad (6.32)$$

if $$p_\Lambda = - c^2 \rho_\Lambda \qquad (6.33)$$

with $$\rho_\Lambda c^2 = \Lambda / \kappa \qquad (6.34)$$

or $$\rho_\Lambda = \frac{\Lambda}{\kappa c^2} = \frac{\Lambda c^2}{8 \pi G} \qquad (6.35)$$

This allows to interpret the Λ-term in Einstein's equations as a special matter component in the otherwise Λ-free equation.
A non-zero Λ would then not be considered as a constant of nature that enters the gravitational law (on the left-hand side) but as the energy-momentum representative of some extreme matter that gives rise to a constant energy density $\rho_\Lambda c^2$ and extreme negative pressure $p_\Lambda = -\rho_\Lambda c^2$ (negative if ρ_Λ positive).
In this interpretation a non-zero Λ requires a further explanation in terms of matter models.
This is the Problem of Dark Energy.

As in the Newtonian case we specify

$$t_0 = \text{"now"} \tag{6.36}$$

and define the Hubble parameter as

$$H(t) = \frac{\dot{a}(t)}{a(t)} . \tag{6.37}$$

In what follows an index "0" indicates evaluation at t_0, e.g.,

$$H_0 = H(t=t_0) = \frac{\dot{a}(t_0)}{a(t_0)} \tag{6.38}$$

We assume

$$\rho = \rho^{rad} + \rho^{dust} \tag{6.39}$$

with

$$(\rho^{rad} a^4) = \text{const}$$
$$= \rho^{rad}_0 a^4_0 \tag{6.40}$$

and $(\rho^{dust} a^3) = \text{const}$

$$= \rho^{dust}_0 a^3_0 \tag{6.41}$$

Hence

$$\rho^{rad}(t) = \rho_0^{rad}\left(\frac{a_0}{a(t)}\right)^4 \tag{6.42}$$

$$\rho^{dust}(t) = \rho_0^{dust}\left(\frac{a_0}{a(t)}\right)^3 \tag{6.43}$$

In analogy to the Newtonian case (Lecture 2, formulae (2.26)) we define the dimensionless parameters Ω, now differentiating between "radiation" and "dust" in the matter component Ω_m:

$$\Omega_{rad} := \frac{8\pi G\rho_0^{rad}}{3H_0^2} \tag{6.44}$$

$$\Omega_{dust} := \frac{8\pi G\rho_0^{dust}}{3H_0^2} \tag{6.45}$$

$$\Omega_k := -\frac{kc^2}{H_0^2 a_0^2} \tag{6.46}$$

$$\Omega_\Lambda := \frac{\Lambda c^2}{3H_0^2} \tag{6.47}$$

where

$$\Omega_m := \Omega_{rad} + \Omega_{dust} \tag{6.48}$$

Using the Ω's the Friedmann equation (6.16b) (i.e. $\dot{a}^2 = \cdots$) is equivalent to (where $\rho = \rho^{rad} + \rho^{dust}$) (6.49)

$$\frac{H^2(t)}{H_0^2} = \Omega_{rad}\left(\frac{a_0}{a(t)}\right)^4 + \Omega_{dust}\left(\frac{a_0}{a(t)}\right)^3 + \Omega_k\left(\frac{a_0}{a(t)}\right)^2 + \Omega_\Lambda \qquad (6.50)$$

Setting again (compare (2.31-32))

$$\lambda := t H_0$$

$$X(\lambda) = a(\lambda/H_0)/a_0$$

we get with

$$\frac{dX}{d\lambda} = \frac{\dot{a}}{a_0 H_0}$$

$$\rightsquigarrow \quad \frac{1}{X}\frac{dX}{d\lambda} = \frac{H}{H_0}$$

and (6.50) is equivalent to

$$\left(\frac{dX}{d\lambda}\right)^2 = \frac{\Omega_{rad}}{X^2} + \frac{\Omega_{dust}}{X} + \Omega_k + \Omega_\Lambda X^2 \qquad (6.52)$$

or

$$\left(\frac{dx}{d\lambda}\right)^2 + V(x) = E \qquad (6.53a)$$

with

$$V(x) = -\left(\frac{\Omega_{rad}}{x^2} + \frac{\Omega_{dust}}{x} + \Omega_\Lambda x^2\right) \qquad (6.53b)$$

$$E = \Omega_k \qquad (6.53c)$$

Evaluating (6.50) at $t = t_0$ results again in the <u>cosmological triangle</u>

$$1 = \underbrace{\Omega_m}_{\Omega_{rad} + \Omega_{dust}} + \Omega_k + \Omega_\Lambda \qquad (6.54)$$

Note that all this is fully analogous to the Newtonian case except for the appearance of Ω_{rad} here which did not exist there. Ω_{rad} dominates the potential $V(x)$ for $x \searrow 0$: it makes it diverge to $-\infty$ more rapidly than Ω_{dust}. Otherwise the discussion very much resembles that already given in Lec. 2.

Hence we have a simple "mechanical" model (potential motion) for $X(x)$, i.e. $a(t)$, depending on Ω_{rad}, Ω_{dust}, Ω_k, and Ω_Λ (only three are independent according to (6.54)). So given the values for the Ω's and an initial value a_0, we can determine $a(t)$, hence $\dot{a}_0$ and therefore H_0 and ρ_0^{rad}, ρ_0^{dust}, k and Λ according to (6.44-48) and then $\rho^{rad}(t)$ and $\rho^{dust}(t)$ through (6.42-43).