Lecture 7: Observable quantities. Redshift, Luminosity Distance and their relation ("Hubble Plot").

We write the FLRW-metric in the form (4.30-31):

$$g = c\,dt \otimes c\,dt - a^2(t)\,\hat{g}_k \tag{7.1a}$$

with

$$\hat{g}_k = d\chi \otimes d\chi + \Sigma_k^2(\chi)\left(d\theta \otimes d\theta + \sin^2\theta\, d\varphi \otimes d\varphi\right) \tag{7.1b}$$

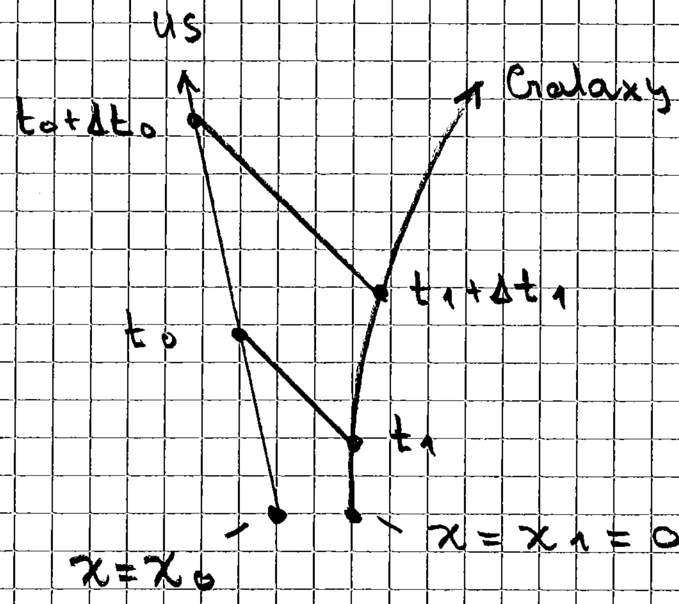
and

$$\Sigma_k(\chi) = \begin{cases} \sin(\chi) & \text{for } k = +1 \\ \chi & \text{for } k = 0 \\ \sinh(\chi) & \text{for } k = -1 \end{cases} \tag{7.1c}$$

If we are at the spatial point $\chi = 0$, any light-ray into the future is given by

$$d\chi = c\,\frac{dt}{a(t)} \tag{7.2}$$

We now consider two co-moving systems, "us" at $\chi = \chi_0$ and a "galaxy" at $\chi = \chi_1 = 0$:



(7.3)

The Galaxy sends light-signals at t_1 and $t_1 + \Delta t_1$, arriving at "us" at times t_0 and $t_0 + \Delta t_0$, respectively. For each of them (7.2) is valid:

$$x_0 = \int_0^{x_0} dx = \int_{t_1}^{t_0} c \frac{dt}{a(t)}$$

$$= \int_{t_1+\Delta t_1}^{t_0+\Delta t_0} c \frac{dt}{a(t)} \qquad (7.4)$$

For $\Delta t_1 \to 0$ and correspondingly $\Delta t_0 \to 0$ we get to first order in Δt for the 2.nd term on r.h.s:

$$\int_{t_1}^{t_0} c \frac{dt}{a(t)} + c \left[\frac{\Delta t_0}{a(t_0)} - \frac{\Delta t_1}{a(t_1)} \right] \qquad (7.5)$$

Equality with l.h.s then gives (writing dt_1 and dt_0 for 1st order)

$$\frac{dt_0}{a(t_0)} = \frac{dt_1}{a(t_1)} \qquad (7.6)$$

From the metric (7.1) it is immediate that along the worldline of a co-moving observer (where $d\chi = d\theta = d\varphi = 0$) dt is the proper length; i.e. (7.6) can also be written as

$$\frac{d\tau_0}{a(t_0)} = \frac{d\tau_1}{a(t_1)} \qquad (7.7)$$

where $d\tau$ is the proper time elapsed along the respective worldlines

If we take t_1 to be the time at which the "Galaxy" starts to send an approximately monochromatic signal to "us" and $t_1 + dt_1$ as the time the "Galaxy" ends sending, then since we receive in our eigen time intervall $d\tau_0$ as many phase-cycles as the "Galaxy" sends off in its eigentime $d\tau_1$, the frequencies ν_0 we receive and ν_1 send off are related by

$$\nu_1 \cdot d\tau_1 = \nu_0 \cdot d\tau_0 \tag{7.8}$$

cycles send off by "Galaxy"

cycles received by "us"

Hence, using (7.7), the redshift factor z obeys

$$z := \frac{\nu_1 - \nu_0}{\nu_0} = \frac{\nu_1}{\nu_0} - 1$$

$$= \frac{d\tau_0}{d\tau_1} - 1$$

$$= \frac{a(t_0)}{a(t_1)} - 1 \tag{7.9}$$

or

$$\boxed{z + 1 = \frac{a(t_0)}{a(t_1)}} \tag{7.10}$$

"Cosm. Redshift"

This formula relates the locally observable redshift to the global property encoded in the scale factor $a(t)$.

For example: If we see the light of a "Galaxy" with redshift z, then it was emitted at a time at which the "size of the universe" $a(t_1)$ was only the $1/(z+1)$ part of that today:

$$a(t_1) = \frac{a(t_0)}{1+z} \tag{7.11}$$

Observed Quasars show redshifts well above 7; eg

$$\text{ULAS J}1342+0928 : \underline{\underline{z = 7.54}} \tag{7.12}$$

↓

UKIDSS - Large - Area - Survey

↓

UKIRT - Infrared - Deep - Sky - Survey

↓

United - Kingdom - Infrared - Telescope
(Mauna - Kea - Obervatory, Hawaii)

So we "see" objects from an expoch at which our universe had less than 1/8 of the "size" today.
That is very early indeed!

Only if the function $a(t)$ is known can we relate z to

a.) the time difference $(t_0 - t_1)$

b.) the geodesic distance

$$d_1 := a(t_1)\,\chi_1$$

to us at the moment of emission.

If we interpreted the redshift as Doppler effect – which is not a good idea in a time dependent space-time geometry ! – we would set

$$1 + z = \frac{\nu_1}{\nu_0} = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} \tag{7.13}$$

where

$$\beta = \frac{v}{c}$$

and v = velocity of recession of the "Galaxy" from us. Hence

$$\beta = \frac{(1+z)^2 - 1}{(1+z)^2 + 1} \tag{7.14}$$

For the $z = 7.54$ Quasar we get

$$\beta = 0.973 \tag{7.15}$$

It is z, not β that is directly observable. Besides, the very notion of "relative velocity" ceases to have a good meaning in a general - in particular non-stationary - Spacetime. We will see this in more detail later.

Similar to "relative velocity" is the fate of the notion of "relative distance". The obvious geometric notion of "simultaneous geodesic distance" exists mathematically, at least in spacetimes with simultaneity structure (i.e. spacelike foliation), but even then is its operational physical realisation elusive. It is therefore replaced by the so-called <u>Luminosity distance</u>

Let L be the absolute brightness – or luminosity – of some source,

$$L = \frac{\text{radiated em-energy}}{\text{time}} \tag{7.16}$$

and l its apparent brightness – or luminosity – at the observer,

$$l = \frac{\text{received em-energy}}{(\text{surface area}) \times (\text{time})} . \tag{7.17}$$

The luminosity distance d_L of the source to the observer is defined by

$$d_L := \left(\frac{L}{4\pi l}\right)^{1/2} \tag{7.18}$$

In flat space have for isotropic emission

$$l = \frac{L}{4\pi r^2} \rightsquigarrow d_L = r \tag{7.19}$$

r = simultaneous metric distance.
In curved spacetimes d_L bears a generally complicated relation to

other, metrically defined distances. This can be derived in FLRW geometries, as we will now see.

In the derivation it is useful to simultaneously use the spatial coordinates (χ, θ, φ), in which the FLRW metric takes the form (4.30)

$$g = c^2 dt^2 - a^2(t)\left\{ d\chi^2 + \Sigma_k^2(\chi)\, d^2\Omega \right\}, \tag{7.20}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$,

or the coordinates (r, θ, φ), in which the FLRW metric reads as in (4.27)

$$g = c^2 dt^2 - a^2(t)\left\{ \frac{dr^2}{1-kr^2} + r^2\, d^2\Omega \right\}. \tag{7.21}$$

The relation is given by

$$r = \Sigma_k(\chi) = \begin{cases} \sin(\chi), & k=1 \\ \chi, & k=0 \\ \sinh(\chi), & k=-1 \end{cases} \tag{7.22}$$

The advantage of the r coordinate is that it is the "areal radius", i.e. that the surface area of a 2-sphere $r = \text{const.}$ at $t = \text{const}$ is

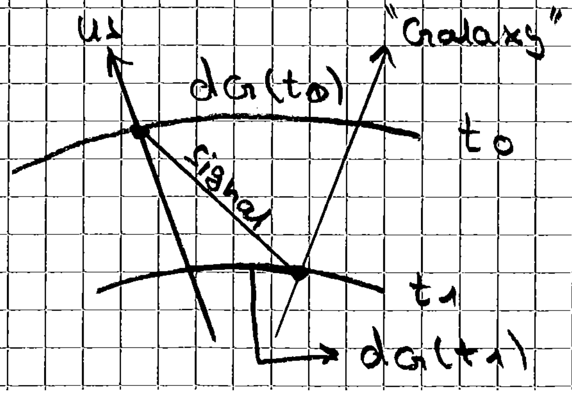
$$A = a^2(t)\, 4\pi r^2 \qquad (7.23)$$

Now, if the source sends out the signal at (r_1, θ, φ), resp. $(\chi_1, \theta, \varphi)$ with $r_1 = \chi_1 = 0$ at time t_1 and we receive the signal at (r, θ, φ) resp. (χ, θ, φ) at time t_0, then the simultaneous distance of the emission- to the reception point at the time t_0 of of emission is

$$d_G(t_1) = a(t_1)\chi \qquad (7.24)$$

and at the time of reception

$$d_G(t_0) = a(t_0)\chi \qquad (7.25)$$



(7.26)

From (7.10) get

$$da(t_0) = (z+1)\, da(t_1) \tag{7.27}$$

The relation to the luminosity distance is as follows: In the (eigen-) time span dt_1 the source emits the energy $E_1 = L\, dt_1$ in form of light, i.e. photons. Each photon reaches us redshifted, i.e. with energy diminished by a factor $\nu_0/\nu_1 = (1+z)^{-1}$. The radiation is assumed to be isotropic, so that the photons have distributed over a surface $4\pi a^2(t_0) r^2$ when they reach us. Hence the amount of energy per surface element and (eigen-) time dt_0 at the observer is

$$l = L\, dt_1 (1+z)^{-1} \frac{1}{4\pi a^2(t_0) r^2} \frac{1}{dt_0}$$

$$= L\, (1+z)^{-2} \frac{1}{4\pi a^2(t_0) r^2} \tag{7.28}$$

From this we get for d_L via (7.18):

$$\boxed{d_L = (1+z)\, a_0\, r} \quad (7.29)$$

Writing from now on

$$\left.\begin{aligned} a_0 &:= a(t_0)\,, \quad t_0 = \text{time of reception} \\ a_1 &:= a(t_1)\,, \quad t_1 = \text{time of emission} \end{aligned}\right\} \quad (7.30)$$

We could replace r by χ :

$$r = \Sigma_k(\chi) = \Sigma_k\left(\frac{a_0 \chi}{a_0}\right) = \Sigma_k\left(\frac{d_G(t_0)}{a_0}\right) \quad (7.31a)$$

$$= \quad '' \quad = \Sigma_k\left(\frac{a_1 \chi}{a_1}\right) = \Sigma_k\left(\frac{d_G(t_1)}{a_1}\right) \quad (7.31b)$$

so that

$$d_L = (1+z)\, a_0\, \Sigma_k\left(\frac{d_G(t_0)}{a_0}\right) \quad (7.32a)$$

$$= (1+z)^2\, a_1\, \Sigma_k\left(\frac{d_G(t_1)}{a_1}\right) \quad (7.32b)$$

In the relevant case of $k = 0$ where $\Sigma_k = \mathrm{id}$ this leads to relations independent of a_0, a_1:

$$d_L = (1+z)\, d_G(t_0) \quad (7.33a)$$

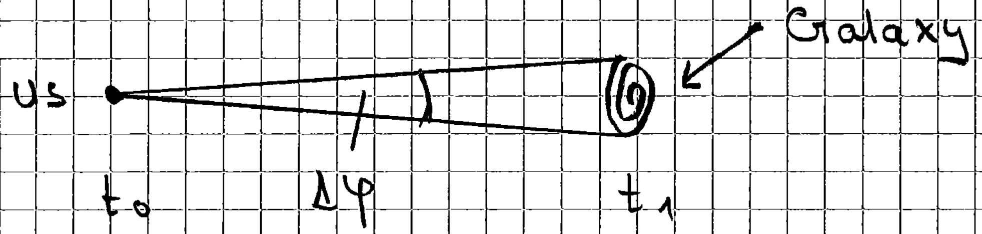
$$= (1+z)^2\, d_G(t_1) \quad (7.33b)$$

for $k = 0$

Other distance measures

Angular distance

Let an observer at $r_0 = \chi_0 = 0$ look at a distant object - like a galaxy - that w.l.o.g. we assume to be in the equatorial plane $\theta = \pi/2$ (i.e. we choose the polar axis appropriately).
The object is located at radial coordinate r or χ. From the outer edges of the object light signals are sent off at time t_1 towards the observer, to be received there at time t_0 with an opening angle $\Delta\varphi$



(7.34)

The angle $\Delta\varphi$ under which we see the transversal extent of the galaxy does not depend on time (for a constant sized galaxy), despite the time dependent metric
This follows from the following

Proposition: The projection of a lightlike geodesic into a hypersurface $t = t_* = \text{const.}$ is, after reparametrisation, a spacelike geodesic with respect to the spatially induced metric.

The proof of this is contained in Problem 4 of Exercise-Sheet 3.

The angle under which the spatial projections of two lightlike geodesics intersect does <u>not</u> depend on t_*, i.e. on which $t = \text{const.}$ hypersurfaces we project, since their metrics only differ by a conformal factor and angles are invariant under conformal transformations.

Let the "true" - i.e. geodesic - spatial size of the source (galaxy) at the time of emission t_1 be D. Since

$$\underbrace{dt}_{t=\text{const}} = \underbrace{dr}_{\text{only transversal size matters}} = \underbrace{d\theta}_{\text{we are in equatorial plane}} = 0 \qquad (7.35)$$

the FLRW metric (7.1) leads to

the spatial metric

$$a^2(t)\,\hat{g}_k =$$

$$ds^2 = a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2\right\} \quad (7.36)$$

$\Rightarrow$

$$D = a(t_1)\, r\, \Delta\varphi \quad (7.37a)$$

$$= \frac{1}{z+1}\, a(t_0)\, r\, \Delta\varphi \quad (7.37b)$$

Now, we define the angular distance d_A as usual by

$$d_A := \frac{D}{\Delta\varphi} = \frac{\text{true extent}}{\text{angle of sight}} \quad (7.38)$$

Hence

$$d_A = \frac{1}{1+z}\, a_0\, r = \frac{d_L}{(1+z)^2} \quad (7.39)$$

where we used (7.29) in last step.

For a flat ($k=0$) universe (7.33b) shows that

$$d_A = d_G^{(1)} \qquad (k=0) \quad (7.40)$$

where we wrote $d_G^{(1)} := d_G(t_1) =$ geodesic distance at time of emission.

Uses of d_L and d_A:

d_L: The luminosity distance can be used with "<u>Standard candles</u>", i.e. objects of <u>known absolute luminosity</u> L. For them we can measure d_L by measuring the (local!) apparent luminosity l:

$$d_L = \left(\frac{L}{4\pi l}\right)^{1/2} \qquad (7.18)$$

Standard candles in cosmology are Cepheid - Stars of variable periodic luminosity, whose maximal brightness is related to its period (period - luminosity - curve). This is due to radial pulsations of the stars atmosphere and correspondingly its temperature, which have been studied in great detail theoretically (e.g. by Eddington). Cepheids are observed up to distances of 20 Mpc $\cong$ $6 \cdot 10^7$ ly by the HST. Beyond such distances supernovae of Type 1A take over as standard candles, giving rise to the <u>cosmological distance ladder</u>

However, one has to take into account possible systematic errors. For example, objects at greater distances are seen at earlier times in the cosmological evolution. This means that they are seen at epochs in the cosmological evolution when there were fewer heavier elements around (fewer "metals", i.e. elements heavier than He). Hence more distant objects have systematically a lower metalicity. But metalicity may influence the period-luminosity-relation of cepheids and also the absolute brightness of a supernovae of type 1A. These systematic effects have to be taken into account in order to not make systematic errors when gauging, say a more distant family of cepheids with a closer one (→ different "populations"). The same holds for SN 1A.

d_A: The angular distance can be used with "standard rulers", i.e. objects of known true size D. For them we can measure d_A by measuring the (local!) apparent size $\Delta\varphi$:

$$d_A = \frac{D}{\Delta\varphi} \qquad (7.38)$$

Such "objects" can also be more abstract, like the microwave background, in which the true size of temperature fluctuations are known by the sound-velocity in plasmas.

Let us collect the distance measures so far:

$d_G^{(0)}$:= geodesic distance at time t_0 of reception

$d_G^{(1)}$:= geodesic distance at time t_1 of emission

d_L := Luminosity distance

d_A := Angular distance.

Only d_L and d_A are locally observable, and only $d_G^{(0)}$ and $d_G^{(1)}$ have an invariant geometric meaning relating to the cosmological model parametrized by Ω_{rad}, Ω_{dust}, Ω_Λ and Ω_k. We will later see that we can express $d_G^{(0)}$ as function of z (redshift) by

$$d_G^{(0)}(z) = \frac{c}{H_0} \times$$

$$\int_0^z \frac{dz'}{\left[(1+z')^4 \Omega_{rad} + (1+z')^3 \Omega_{dust} + (1+z')^2 \Omega_k + \Omega_\Lambda\right]^{1/2}} \qquad (7.41)$$

Lets look at a simple case :

$$\left.\begin{aligned} &\Omega_{rad} = \Omega_k = \Omega_\Lambda = 0 \\ &\Omega_{dm} = 1 \end{aligned}\right\} \quad (7.42)$$

A matter-dominated ($\Omega_{dm} \gg \Omega_{rad}$), flat ($\Omega_k = 0$) universe without cosmological constant ($\Omega_\Lambda = 0$). Then

$$\frac{c}{H_0} \int_0^z \frac{dz'}{(1+z')^{3/2}} = -\frac{2c}{H_0} (1+z')^{-1/2} \Big|_0^z$$

$$= \frac{2c}{H_0} \left(1 - \frac{1}{\sqrt{1+z}} \right)$$

$$= d_G^{(0)}(z) \qquad (7.43)$$

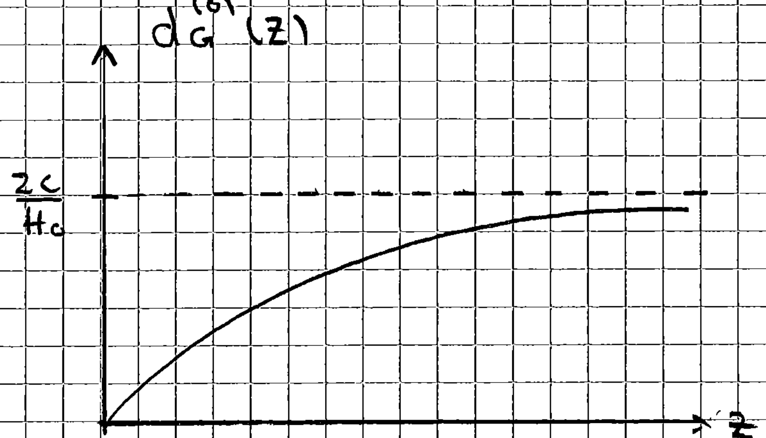
For $k=0$ we can infer d_L and d_A from (7.33a) and (7.39), respectively :

$$d_L(z) = \frac{2c}{H_0} \left(1+z - \sqrt{1+z} \right) \qquad (7.44)$$

$$d_A(z) = \frac{2c}{H_0} \left(\frac{1}{1+z} - \frac{1}{(1+z)^{3/2}} \right) \qquad (7.45)$$

From $d'^{(0)}_G(z) = \frac{c}{H_0} (1+z)^{-3/2} > 0$

the graph of $d^{(0)}_G$ looks like



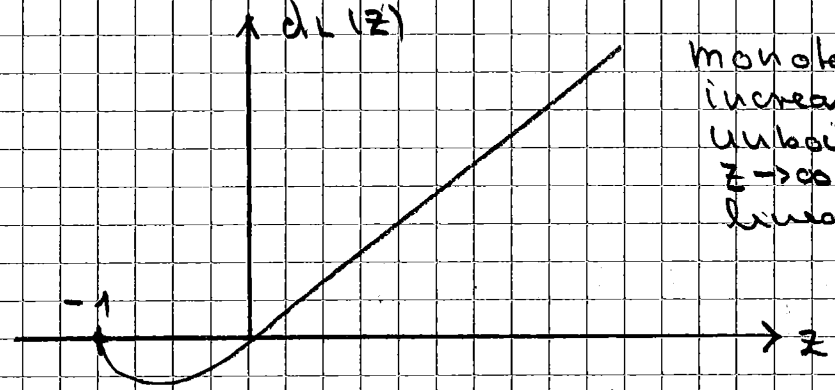
monotonically increasing and bounded for $z \to \infty$ by $\frac{2c}{H_0}$ (7.46)

From $d_L'(z) = \frac{2c}{H_0}\left(1 - \frac{1}{2}(1+z)^{-1/2}\right)$ which vanishes iff

$$d_L'(z) = 0 \iff z = -\frac{3}{4} < 0$$

and $d_L''(z) = \frac{c}{2H_0}(1+z)^{-3/2} > 0$.

Hence $z = -\frac{3}{4}$ is a local minimum in the unphysical regime $z < 0$. For $z \geq 0$ d_L is monotonically increasing



monotonically increasing and unbounded for $z \to \infty$; asympt. linear. (7.47)

Finally, $d_A(z)$ has a single zero at $z = 0$, is positive for $z > 0$ and approaches zero for $z \to \infty$. Have

$$d_A'(z) = \frac{2c}{H_0}(1+z)^{-5/2}\left[\frac{3}{2} - \sqrt{1+z}\right]$$

$$d_A''(z) = \frac{4c}{H_0}(1+z)^{-7/2}\left[\sqrt{1+z} - \frac{15}{8}\right]$$

Hence $d_A'(z) = 0 \Leftrightarrow z = 5/4$ and

$d''_A(5/4) = \sim [3/2 - 15/8] \sim -3/8 < 0$,

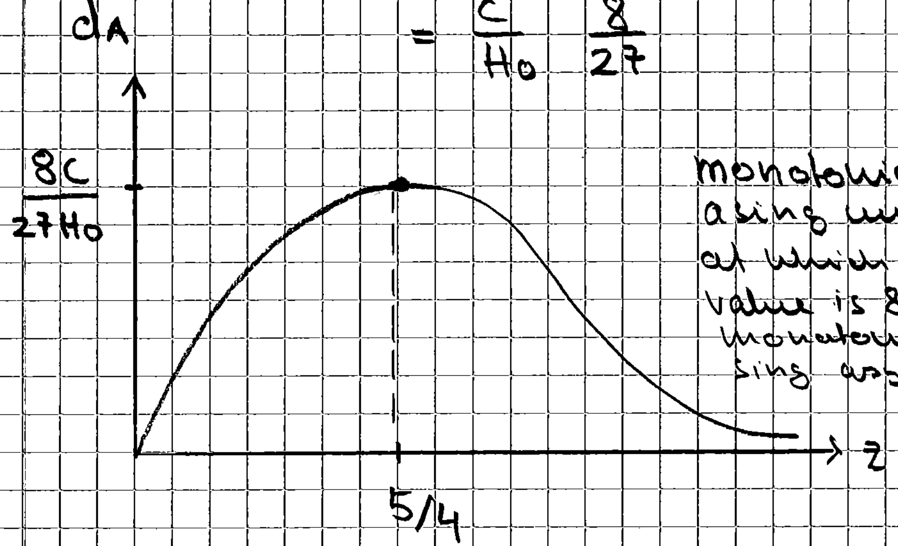
so that there is a single maximum

at $z = z_{max} = 5/4$, $\sqrt{1+z_{max}} = 3/2$

$$d_A(z_{max}) = \frac{2c}{H_0}\left[\frac{1}{1+z_{max}} - \frac{1}{(1+z_{max})^{3/2}}\right]$$

$$= \frac{2c}{H_0}\left[\frac{4}{9} - \frac{8}{27}\right]$$

$$= \frac{c}{H_0}\frac{8}{27} \qquad (7.48)$$



monotonically increasing until $z = 1.25$, at which maximal value is $8c/27H_0$, then monotonically decreasing asymptotically to zero for $z \to \infty$ (7.49)

Note that for $k = 0$ we have $d_A = d_G^{(1)}$. Hence the plot just shown is also valid for $d_G^{(1)}$, which looks quite different from that of $d_G^{(0)}$. Note that $d_G^{(1)}(z \to \infty) \to 0$ is clear, as we look back to an earlier and earlier time where $a \to 0$ and hence metric distances tend to zero.

[(7.49) explains "Mr Tur Tur", the illusionary giant ("Scheinriese") from Jim Knopf und Lukas der Lokomotivführer.]

Having looked at a special case ($\Omega_{dust} = 1$) we are now interested in the general case, in which we also like to express distances - and in particular d_L - as function of z.

The first thing we notice is that (6.50)

$$\left(\frac{H(t)}{H_0}\right)^2 = \Omega_{rad}\left(\frac{a_0}{a}\right)^4 + \Omega_{dust}\left(\frac{a_0}{a}\right)^3 + \Omega_k\,(a_0/a)^2 + \Omega_\Lambda$$

can be written in terms of redshift (7.10)

$$z+1 = \frac{a_0}{a}$$

Hence

$$\boxed{\left[\frac{H(z)}{H_0}\right]^2 = \Omega_{rad}(z+1)^4 + \Omega_{dust}(z+1)^3 + \Omega_k(z+1)^2 + \Omega_\Lambda} \qquad (7.50)$$

Next we note that dt can be expressed by dz:

$$1+z = \frac{a_0}{a(t)}$$

$$\rightsquigarrow \quad dz = -\frac{a_0}{a^2(t)}\,\dot{a}(t)\,dt$$

$$= -\frac{a_0}{a(t)}\,\frac{\dot{a}(t)}{a(t)}\,dt$$

$$= -(1+z)\,H(t)\,dt$$

$$\rightsquigarrow \quad \boxed{dt = -\frac{1}{H(z)}\,\frac{dz}{1+z}} \qquad (7.51)$$

where $H(z)$ is as in (7.50).

Equation (7.51) tells how an increment dz in redshift is related to an increment $-dt$ in look-back time.

The total "look-back-time" $T(z)$ from $z=0$ (us) to some value z corresponding to the source is then obtained from integrating (7.51). Noting that

$$\int_{t_0}^{t_1} dt = (t_1 - t_0) = -\int_0^z dz\,(\ldots)$$

we get

$$\boxed{T(z) = t_0 - t_1 = \int_0^z \frac{dz'}{(1+z')H(z')}} \tag{7.52}$$

"Look-Back-Time"

The "look-back-distance" is just $d_C^{(1)}(z)$ and follows similarly: For an incoming lightlike geodesic, i.e. starting at (t_1, χ) ending at $(t_0, \chi=0)$, we have

$$c\,dt = \underset{\substack{\uparrow \\ \text{incoming}}}{-}\, a(t)\, d\chi \tag{7.53}$$

hence

$$d\chi = -c\,\frac{dt}{a(t)} = -c\,\frac{da}{\dot{a}\,a}$$

$$= -c\,\frac{da}{H\,a^2}$$

$$= \frac{c}{a_0}\,\frac{dz}{H(z)} \tag{7.54}$$

Where in the last step we used

$$dz = -\frac{a_0}{a^2}\,da \Rightarrow \frac{da}{a^2} = -\frac{dz}{a_0} \tag{7.55}$$

Integrating (7.54) and multiplying with a_0 gives $d_G^{(0)}(z)$

$$-d_G^{(0)}(z) = a_0 \int_{\chi_1 = \chi}^{\chi_0 = 0} d\chi = c \int_{z_1 = z}^{z_0 = 0} \frac{dz'}{H(z')}$$

or

$$\boxed{d_G^{(0)}(z) = c \int_0^z \frac{dz'}{H(z')}} \qquad (7.56)$$

Equations (7.32a) and (7.39) thus allow to express d_L and d_A as functions of z:

$$d_L(z) = (1+z)\, a_0 \Sigma_K \left(\frac{d_G^{(0)}(z)}{a_0} \right) \qquad (7.57a)$$

$$d_A(z) = \frac{1}{1+z}\, a_0 \Sigma_K \left(\frac{d_G^{(0)}(z)}{a_0} \right) \qquad (7.57b)$$

$\Rightarrow$ Luminosity and angular distance as function of a_0, H_0, all Ω's, and z.

Note that in (r, θ, φ)-coordinates we have from (7.31a):

$$a_0 r = a_0 \Sigma_K \left(d_G^{(0)}(z)/a_0 \right) \qquad (7.58)$$

Example: Mattig's Formula

In case of vanishing cosmological constant $d_G^{(0)}(z)$ can be explicitly integrated from (7.56). This was first done by Wolfgang Mattig in 1957, also assuming $\Omega_{rad} = 0$.

We generally have from (7.50)

$$H(z) = H_0 \sqrt{h(z)} \quad (7.59)$$

$$h(z) := \Omega_{rad}(z+1)^4 + \Omega_{dust}(z+1)^3 + \Omega_k (z+1)^2 + \Omega_\Lambda \quad (7.60)$$

For $\Omega_\Lambda = \Omega_{rad} = 0$, so that the cosmological triangle (6.54) implies

$$\Omega_k = 1 - \Omega_{dust} \quad (7.61)$$

we have

$$h(z) = (z+1)^2 \left[\Omega (z+1) + (1-\Omega) \right] \quad (7.62)$$

where for simplicity we wrote

$$\Omega := \Omega_{dust}. \quad (7.63)$$

Equation (7.56) now reads

$$d_G^{(0)}(z) = \frac{c}{H_0}\int_0^z \frac{dz'}{(z'+1)\left[\Omega\,(z'+1)+(1-\Omega)\right]^{1/2}} \qquad (7.64)$$

We substitute

$$u = \frac{1}{z'+1} \;\curvearrowright\; \frac{dz'}{z'+1} = -\frac{du}{u} \qquad (7.65)$$

$$d_G^{(0)}(z) = \frac{c}{H_0}\int_{\frac{1}{z+1}}^{1} \frac{du}{\left[(1-\Omega)u^2+\Omega\,u\right]^{1/2}} \qquad (7.66)$$

At this point we need to make an assumption about the sign of the highest-power coefficient $(1-\Omega)=\Omega_K$ under the square-root. Here we shall assume

$$\Omega - 1 = -\Omega_K = \frac{kc^2}{H_0^2 a_0^2} > 0 \qquad (7.67)$$

i.e. positive curvature. The other cases will be dealt with in the exercises, sheet 4. Equation (7.66) can now be written, using

$(1-\Omega)u^2 + \Omega\,u =$

$(\Omega-1)\left[-u^2 + \frac{\Omega}{\Omega-1}u\right]$

$= (\Omega-1)\left[-\left(u-\frac{\Omega}{2(\Omega-1)}\right)^2 + \left(\frac{\Omega}{2(\Omega-1)}\right)^2\right]$

$$d_G^{(0)}(z) = \frac{c}{H_0\sqrt{\Omega-1}} \times \int_{\frac{1}{1+z}}^{1} \frac{d\mu}{\left[-\left(\mu - \frac{\Omega}{2(\Omega-1)}\right)^2 + \left(\frac{\Omega}{2(\Omega-1)}\right)^2\right]^{1/2}} \qquad (7.68)$$

The prefactor is, using (6.46),

$$\Omega - 1 = -\Omega_k = \frac{k c^2}{H_0^2 a_0^2}, \qquad (7.69)$$

hence, with our choice $k = +1$,

$$\frac{c}{H_0\sqrt{\Omega-1}} = a_0 . \qquad (7.70)$$

Using the variable

$$y := \mu - \frac{\Omega}{2(\Omega-1)} \qquad (7.71)$$

(7.68) becomes

$$d_G^{(0)}(z) = a_0 \int_{\frac{1}{1+z} - \frac{\Omega}{2(\Omega-1)}}^{1 - \frac{\Omega}{2(\Omega-1)}} \frac{dy}{\left[-y^2 + \left(\frac{\Omega}{2(\Omega-1)}\right)^2\right]^{1/2}}$$

$$= a_0 \arcsin\left(\frac{2(\Omega-1)y}{\Omega}\right)\Bigg|_{\frac{1}{1+z} - \frac{\Omega}{2(\Omega-1)}}^{1 - \frac{\Omega}{2(\Omega-1)}}$$

$$= a_0 \Big[\operatorname{arsin}\left(\frac{\Omega-2}{\Omega}\right)$$

$$- \operatorname{arsin}\left(\frac{2(\Omega-1)}{(1+z)\Omega} - 1\right)\Big] \qquad (7.72)$$

The arsin - functions obey the "addition law"

$$\operatorname{arsin}(a) - \operatorname{arsin}(b) =$$

$$\operatorname{arsin}\left(a\sqrt{1-b^2} - b\sqrt{1-a^2}\right) \qquad (7.73)$$

which follows from

$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha)$$

$$\alpha - \beta = \operatorname{arsin}\left(\sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha)\right)$$

$$\alpha = \operatorname{arsin}(a) \qquad \beta = \operatorname{arsin}(b).$$

In our case

$$a = \frac{\Omega-2}{\Omega}, \quad b = \frac{2(\Omega-1)}{(1+z)\Omega} - 1 \qquad (7.74)$$

$$\curvearrowright \quad a\sqrt{1-b^2} - b\sqrt{1-a^2} =$$

$$\frac{\Omega-2}{\Omega}\left(1 - \left(\frac{\Omega - z\Omega - 2}{(1+z)\Omega}\right)^2\right)^{1/2}$$

$$- \frac{\Omega - z\Omega - 2}{(1+z)\Omega}\left(1 - \left(\frac{\Omega-2}{\Omega}\right)^2\right)^{1/2} \qquad (7.75)$$

$$a\sqrt{1-b^2} - b\sqrt{1-a^2}$$

$$= \frac{\Omega-2}{\Omega^2(1+z)}\left[(1+z)^2\Omega^2 - (\Omega - z\Omega - 2)^2\right]^{1/2}$$

$$- \frac{\Omega - z\Omega - 2}{(1+z)\Omega^2}\underbrace{\left[\Omega^2 - (\Omega-2)^2\right]}_{4(\Omega-1)}^{1/2} \qquad (7.76)$$

The first square-bracket is

$$\Omega^2(1+z)^2 - (\Omega - z\Omega - 2)^2$$

$$= \cancel{\Omega^2 z^2} + \cancel{\Omega^2} + \underline{2z\Omega^2} - \cancel{\Omega^2} - \cancel{z^2\Omega^2} - 4$$
$$+ \underline{2z\Omega^2} + 4\Omega - \underline{4z\Omega}$$

$$= 4z\Omega^2 - 4z\Omega - 4(1-\Omega)$$

$$= 4z\Omega(\Omega-1) + 4(\Omega-1)$$

$$= 4(\Omega-1)(1+z\Omega) \qquad (7.77)$$

Hence

$$a\sqrt{1-b^2} - b\sqrt{1-a^2}$$

$$= 2(\Omega-1)^{1/2}\left\{\frac{\Omega-2}{\Omega^2(1+z)}(1+z\Omega)^{1/2} - \frac{\Omega - z\Omega - 2}{(1+z)\Omega^2}\right\}$$

$$= 2\frac{(\Omega-1)^{1/2}}{\Omega^2(1+z)}\left\{(\Omega-2)\left((1+z\Omega)^{1/2}-1\right) + z\Omega\right\}$$

$$= \frac{2c}{H_0 a_0}\,\frac{(\Omega-2)\left((1+z\Omega)^{1/2}-1\right) + z\Omega}{\Omega^2(1+z)} \qquad (7.78)$$

where we used once more (7.69), i.e.

$$(\Omega - 1)^{1/2} = \frac{c}{H_0 a_0}.$$

Hence we have evaluated the right-hand side for (7.68)

$$d_a^{(0)}(z) = a_0 \arcsin\left\{ \frac{2c}{H_0 a_0} \frac{(\Omega-2)\left((1+z\Omega)^{1/2}-1\right)+z\Omega}{\Omega^2(1+z)} \right\}$$

$$= a_0 \sum\nolimits_k^{-1} \left(\frac{2c}{H_0 a_0} \frac{(\Omega-2)\left((1+z\Omega)^{1/2}-1\right)+z\Omega}{\Omega^2(1+z)} \right) \qquad (7.79)$$

Now, look at the expressions (7.57) for d_L and d_A: There $\sum_k$ and $1/a_0$ appears precisely in the term so as to cancel them in (7.79)!

Hence

$$\boxed{d_L(z) = \frac{2c}{H_0} \frac{(\Omega-2)\left[(1+z\Omega)^{1/2}-1\right]+z\Omega}{\Omega^2}} \qquad (7.80a)$$

$$\boxed{d_A(z) = \frac{2c}{H_0} \frac{(\Omega-2)\left[(1+z\Omega)^{1/2}-1\right]+z\Omega}{(1+z)^2\,\Omega^2}} \qquad (7.80b)$$

"Mattig's Formulae"

(valid for $k = \pm 1$ and $= 0$)

What Mattig actually proved was

$$a_0 r(z) = \frac{2c}{H_0} \frac{(\Omega-2)\left(\sqrt{1+z\Omega}-1\right)+z\Omega}{\Omega^2 (1+z)} \qquad (7.81)$$

which in view of (7.29) is just (7.80a).

It is quite surprising that k, i.e. Σ_k, and a_0 do not enter (7.80). The graph $d_L(z)$ only depends on Ω and H_0.

Definition: The graph $z \to d_L(z)$ is called the <u>Hubble plot</u>

Unfortunately no such simple analytic formula exists for $\Lambda \neq 0$. As current values of Ω_Λ are approximately twice as large as $\Omega = \Omega_{dust}$, the above formulae are not realistic.

In the general case we can give at least a power-series expansion for the Hubble plot. For that we express r as power series in z and then, via (7.29), we get $d_L(z)$.

Hubble Plot: Power Series

For light propagation we have for an incomming ray in (t, r, θ, φ) coordinates at $r = 0$ (observer)

$$c\,dt = -\,a(t)\,\frac{dr}{\sqrt{1-kr^2}}$$

$$c\int_{t_1}^{t_0}\frac{dt}{a(t)} = -\int_{r}^{0}\frac{dr'}{\sqrt{1-kr'^2}}$$

$$= \int_{0}^{r}\frac{dr'}{\sqrt{1-kr'^2}} \qquad (7.82)$$

and

$$z = \frac{a(t_0)}{a(t_1)} - 1 \qquad (7.83)$$

Now, we expand $a(t)$ at $t = t_0$:

$$a(t_1) = a(t_0) + (t_1 - t_0)\,\dot{a}(t_0) + \frac{1}{2}(t_1 - t_0)^2\,\ddot{a}(t_0) + \cdots$$

$$= a(t_0)\left[1 - (t_0 - t_1)\,H_0 - \frac{1}{2}\,q_0 H_0^2 (t_0 - t_1)^2 + \cdots\right] \qquad (7.84)$$

Where $\qquad H_0 := \dot{a}(t_0)\,/\,a(t_0)$

$$q_0 := -\,\ddot{a}(t_0)\,a(t_0)\,/\,\dot{a}^2(t_0)$$

Inserting this into the expression (7.83) for Z we get to second order in $(t_0 - t_1)$:

$$Z = \left[1 - (t_0 - t_1)H_0 - \tfrac{1}{2} q_0 H_0^2 (t_0 - t_1)^2\right]^{-1} - 1$$

$$= 1 + H_0 (t_0 - t_1) + H_0^2 (t_0 - t_1)^2 + \tfrac{1}{2} q_0 H_0^2 (t_0 - t_1)^2 - 1$$

$$= H_0 (t_0 - t_1) + \left(1 + \frac{q_0}{2}\right) H_0^2 (t_0 - t_1)^2 \qquad (7.85)$$

The inverse of that is to the same order

$$(t_0 - t_1) = H_0^{-1}\left[Z - \left(1 + \frac{q_0}{2}\right) Z^2\right] \qquad (7.86)$$

An approximate evaluation of (7.82) gives

$$\int_0^r \frac{dr'}{\sqrt{1 - k r'^2}} = r + O(r^3)$$

$$c \int_{t_1}^{t_0} \frac{dt}{a(t)} = \frac{c}{a(t_0)} \int_{t_1}^{t_0} \frac{dt}{1 + (t - t_0) H_0 + \cdots}$$

$$= \frac{c}{a_0}\left[(t_0 - t_1) + \tfrac{1}{2}(t_0 - t_1)^2 H_0 + \cdots\right] \qquad (7.87)$$

Into this we insert (7.86) and calculate $r(Z)$ to second order:

$$r = \frac{c}{a_0 H_0}\left[z - \left(1 + \frac{q_0}{2}\right) z^2 + \frac{1}{2} z^2 \right]$$

$$= \frac{c}{a_0 H_0}\left[z - \frac{1}{2}(1+q_0) z^2 + O(z^3)\right] \quad (7.88)$$

$$\Rightarrow \quad a_0 r(z) = \frac{c}{H_0}\left[z - \frac{1}{2}(1+q_0) z^2 + O(z^3)\right] \quad (7.89)$$

This gives for d_L via (7.29)

$$\boxed{\begin{aligned} d_L(z) &= (1+z)\, a_0 r \\ &= \frac{c}{H_0}\left[z + \frac{1}{2}(1-q_0) z^2 + O(z^3)\right] \end{aligned}} \quad (7.90)$$

For the angular distance we get

$$\boxed{\begin{aligned} d_A(z) &= (1+z)^{-1}\, a_0 r \\ &= \frac{c}{H_0}\left[z - \frac{1}{2}(3+q_0) z^2 + O(z^3)\right] \end{aligned}} \quad (7.91)$$

This shows that the linear term in the Hubble plot $d(z)$ determines H_0 whereas the quadratic correction leads to q_0.

In order to relate this to the parameters of the model, i.e. the Ω's, we have to relate q_0 to H_0 and the Ω's. For this we use the first Friedmann equation (6.16a):

$$\ddot{a} = -\frac{4\pi G}{3} a \left(\varrho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} a \qquad (7.92)$$

$$\rightsquigarrow q(t) = -\frac{\ddot{a}\,a}{\dot{a}^2} = \frac{4\pi G}{3} H^{-2}\left(\varrho + \frac{3p}{c^2}\right) - \frac{\Lambda c^2}{3} H^{-2} \qquad (7.93)$$

or

$$q_0 = \frac{4\pi G}{3} H_0^{-2}\left(\varrho_0 + \frac{3p_0}{c^2}\right) - \frac{\Lambda c^2}{3} H_0^{-2} \qquad (7.94)$$

For radiation + dust have

$$\varrho_0 = \varrho_0^{rad} + \varrho_0^{dust} \qquad (7.95)$$

$$p_0 = p_0^{rad} = \frac{1}{3}\varrho_0^{rad} c^2 \qquad (7.96)$$

$$\rightsquigarrow \varrho_0^{rad} + \frac{3p_0^{rad}}{c^2} = 2\varrho_0^{rad} \qquad (7.97)$$

$$\Rightarrow q_0 = \frac{4\pi G}{3} H_0^{-2}\varrho_0^{dust} + \frac{8\pi G}{3} H_0^{-2}\varrho_0^{rad} - \frac{\Lambda c^2}{3} H_0^{-2} \qquad (7.98)$$

Using the definition of the Ω's (6.44–47) we see

$$\boxed{q_0 = \frac{1}{2}\Omega_{dust} + \Omega_{rad} - \Omega_\Lambda} \tag{7.99}$$

In the matter-dominated phase where $\Omega_{rad} \ll \Omega_{dust} = \Omega_m$ (m for "matter"), this becomes

$$q_0 = \frac{1}{2}\Omega_m - \Omega_\Lambda \tag{7.100}$$