Lecture 8: Age of universe

galaxy counts, electromagnetic radiation

In Lecture 7 we derived the following expression for the "Look-Back-Time" (see formula (7.52)):

$$T(z) = \int_0^z \frac{dz'}{(1+z')\,H(z')} \tag{8.1}$$

where

$$H(z) = H_0 \sqrt{h(z)} \tag{8.2}$$

and

$$h(z) = \Omega_{rad}\,(z+1)^4 + \Omega_{dust}\,(z+1)^3 + \Omega_k\,(z+1)^2 + \Omega_\Lambda \tag{8.3}$$

From (8.1) we see that $\lim_{z\to\infty} T(z)$ exists, i.e. the integral converges for upper boundary $z \to \infty$ if in (8.3) there is a highest power of z in $h(z)$ greater than zero with positive coefficient, e.g. $\Omega_{dust} > 0$.
In this case we define the Age of the Universe, t_0, by $T(\infty)$:

$$t_0 := \int_0^\infty \frac{dz}{(1+z)\,H(z)}$$

$$= \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)\sqrt{h(z)}} \tag{8.4}$$

If we neglect Ω_{rad} and write $\Omega_{dust} = \Omega_m$, where

$$\Omega_m + \Omega_k + \Omega_\Lambda = 1 \tag{8.5}$$

We get for $h(z)$ in term of Ω_m, Ω_Λ:

$$h(z) = (1+z)^3\,\Omega_m + (1+z)^2\,(1-\Omega_m-\Omega_\Lambda) + \Omega_\Lambda$$

$$= (1+z)^2\,[1+z\,\Omega_m] - z\,(z+2)\,\Omega_\Lambda \tag{8.6}$$

Hence

$$t_0 = \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)\sqrt{(1+z)^2\,[1+z\Omega_m] - z(z+2)\Omega_\Lambda}} \tag{8.7}$$

The polynomial $h(z)$ is, explicitly expanded,

$$h(x) = z^3\,\Omega_m + z^2\,(1+2\,\Omega_m - \Omega_\Lambda)$$

$$+ z\,(2+\Omega_m - 2\Omega_\Lambda) + 1$$

$$= z^3\,\Omega_m + z^2\,(3\Omega_m - \Omega_k)$$

$$+ z\,(3\,\Omega_m - 2\,\Omega_k) + 1 \tag{8.8}$$

Showing that for $\Omega_m > 0$ and $|\Omega_k| \ll \Omega_m$ all coefficients are positive

An often quoted approximation for (8.7) in the range

$$\left.\begin{aligned} &0.1 < \Omega_m < 1 \\ \text{and} \quad &|\Omega_\Lambda| \leq 1 \end{aligned}\right\} \tag{8.9}$$

is

$$\boxed{t_0 = \frac{1}{H_0} \cdot \frac{2}{3} \cdot \left[0.7\,\Omega_m - 0.3\,\Omega_\Lambda + 0.3\right]^{-0.3}} \tag{8.10}$$

$$\left.\begin{aligned} \text{For} \quad \Omega_m &= 0.27 \\ \Omega_\Lambda &= 0.73 \end{aligned}\right\} \tag{8.11}$$

this gives

$$t_0 = 0.988\ \frac{1}{H_0} \tag{8.12}$$

Compare this to solution to Problem 5 of Sheet 4.

Geometry of galaxy-count

A homogeneous distribution gives numbers proportional to volume. But volume grows with radius in a manner revealing the curvature. Hence galaxy-counts may give access to Ω_k or $\Omega_m + \Omega_\Lambda$.

The FLRW-metric is

$$g = c\,dt \otimes c\,dt - a^2(t)\,\hat{g} \tag{8.13}$$

and in (r, θ, φ) coordinates

$$\hat{g} = \frac{dr \otimes dr}{1 - k r^2} + r^2 (d\theta \otimes d\theta + \sin^2\theta\, d\varphi \otimes d\varphi) \tag{8.14}$$

The spatial volume element in the $t = \text{const}$ restriction of g, ie. $a^2 \hat{g}$, is

$$dV = \frac{a^3(t)\, r^2}{(1 - k r^2)^{1/2}}\, dr \wedge d\Omega \tag{8.15}$$

where

$$d\Omega = \sin\theta\, d\theta \wedge d\varphi \tag{8.16}$$

We do not directly have access to r but rather measure the distance through its redshift z (from which we may calculate $d_{cr}^{(o)}$, d_L, d_A).

Along an incomming lightlike geodesic we have

$$\frac{dr}{(1-kr^2)^{1/2}} = -c\,\frac{dt}{a(t)}$$

$$= c\,\frac{dz}{a_0 H(z)} \qquad (8.17)$$

since $\frac{dt}{a(t)} = \frac{da}{\dot{a}\,a} = \frac{da}{H a^2}$ (8.18)

and $1+z = \frac{a_0}{a} \rightsquigarrow dz = -\frac{a_0\,da}{a^2}$ (8.19)

therefore

$$\frac{dt}{a(t)} = -\frac{dz}{a_0 H(z)}. \qquad (8.20)$$

(We already used this in Lecture 7, with χ instead of r, compare (7.54))

Hence in (8.15) we can replace

$\frac{dr}{(1-kr^2)^{1/2}}$ by (8.17) and

$$a^3(t) = \frac{a_0^3}{(1+z)^3} \tag{8.21}$$

Hence

$$dV = \frac{a_0^3}{(1+z)^3} \, r^2(z) \, \frac{c}{a_0 H(z)} \, dz \wedge d\Omega$$

$$= c \, \frac{a_0^2 \, r^2(z)}{(1+z)^3 \, H(z)} \, dz \wedge d\Omega \tag{8.22}$$

But, according to (7.31a) we have

$$r(z) = \Sigma_K \left(d_a^{(0)}(z) / a_0 \right) \tag{8.23}$$

and according to (7.32a)

$$d_L = (1+z) \, a_0 \, \Sigma_K \left(d_a^{(0)}(z) / a_0 \right)$$

$$= (1+z) \, a_0 \, r(z) \, . \tag{8.24}$$

Therefore

$$dV = \frac{c}{H(z)} \, \frac{d_L^2(z)}{(1+z)^5} \, dz \wedge d\Omega$$

$$= \frac{c}{H_0} \cdot \frac{H_0}{H(z)} \cdot \frac{d_L^2(z)}{(1+z)^5} \, dz \wedge d\Omega \tag{8.25}$$

Example:

$$\Omega_{rad} = \Omega_{\Lambda} = 0 \qquad (8.26)$$

$$\Rightarrow \left.\begin{aligned} \Omega_{dun} &= \Omega_m =: \Omega \\ \Omega_k &= 1-\Omega \end{aligned}\right\} \qquad (8.27)$$

$$\begin{aligned} H(z) &= H_0\left[(1+z)^3\Omega + (1+z)^2(1-\Omega)\right]^{1/2} \\ &= H_0(1+z)\left[(1+z)\Omega + (1-\Omega)\right]^{1/2} \\ &= H_0(1+z)\left[1+z\Omega\right]^{1/2} \qquad (8.28) \end{aligned}$$

And by Mattig's formula (7.80a)

$$d_L(z) = \frac{2c}{H_0}\frac{(\Omega-2)\left[(1+z\Omega)^{1/2}-1\right]+z\Omega}{\Omega^2} \qquad (8.29a)$$

$$= \frac{c}{H_0}\cdot z\cdot\frac{1+z+\sqrt{1+z\Omega}}{1+\frac{1}{2}z\Omega+\sqrt{1+z\Omega}} \qquad (8.29b)$$

where the second expression follows from Problem 1 of Sheet 4.

Hence

$$dV = \frac{c}{H_0}\frac{1}{(1+z)(1+z\Omega)^{1/2}}\,\frac{1}{(1+z)^5}$$

$$\times\left(\frac{c}{H_0}\right)^2 z^2\left[\frac{1+\sqrt{1+z\Omega}+z}{1+\sqrt{1+z\Omega}+\frac{1}{2}z\Omega}\right]^2 \qquad (8.30)$$

$$dV = \left(\frac{c}{H_0}\right)^3 \cdot \frac{1}{(1+z)^6} \cdot \frac{1}{\sqrt{1+z\Omega}} \cdot$$

$$\cdot \left[\frac{1+\sqrt{1+z\Omega}+z}{1+\sqrt{1+z\Omega}+\frac{1}{2}\Omega z}\right]^2 \cdot z^2 \; dz \wedge d\Omega \qquad (8.31)$$

Expanding the three terms for $z \ll 1$ gives to linear order in z

$$\frac{1}{(1+z)^6} \cdot \frac{1}{\sqrt{1+z\Omega}} \left\{\frac{1+(1+z\Omega)^{1/2}+z}{1+(1+z\Omega)^{1/2}+\frac{1}{2}\Omega z}\right\}^2$$

$$(1-6z)\left(1-\frac{z\Omega}{2}\right) \cdot \left[\frac{2+\frac{1}{2}z\Omega+z}{2+\frac{1}{2}z\Omega+\frac{1}{2}z\Omega}\right]^2$$

$$= (1-6z)\left(1-\frac{z\Omega}{2}\right)\left[\left(1+\frac{1}{4}z\Omega+\frac{1}{2}z\right)\left(1-\frac{1}{2}z\Omega\right)\right]^2$$

$$= \left(1-6z-\frac{z\Omega}{2}\right)\left[\left(1+\frac{1}{4}z\Omega+\frac{1}{2}z-\frac{1}{2}z\Omega\right)\right]^2$$

$$= \left(1-6z-\frac{z\Omega}{2}-\frac{1}{2}z\Omega+z\right)$$

$$= 1-5z-z\Omega = 1-z(5+\Omega) \qquad (8.32)$$

Hence

$$dV = \left(\frac{c}{H_0}\right)^3 \left(z^2-z^3(5+\Omega)\right) \; dz \wedge d\Omega$$

$$\int dV = \left(\frac{c}{H_0}\right)^3 \left(\frac{4\pi}{3}z^3 - \frac{\pi}{4}z^4(\Omega+5)\right) \qquad (8.33)$$

Planck-Radiation

In thermodynamic equilibrium the spectral distribution of electromagnetic radiation at temperature T is given by

$$\mathcal{E}_\nu \, d\nu = h\nu \, \frac{8\pi}{c^3} \, \frac{\nu^2 d\nu}{\exp(h\nu/kT)-1} \qquad (8.34)$$

(energy per unit volume within frequency interval $[\nu, \nu+d\nu]$)

Hence the number of photons per unit volume and within frequency intervall $[\nu, \nu+d\nu]$ is

$$n_\nu \, d\nu = \frac{8\pi}{c^3} \, \frac{\nu^2 d\nu}{\exp(h\nu/kT)-1} \qquad (8.35)$$

Let this radiation be in thermodynamic equilibrium at time $t = t_1$ with temperature T_1, filling the universe homogeneously. We ask for the spectral distribution at time $t = t_0 > t_1$. The expansion will alter the distribution and we cannot be sure that it stays Planckian.

We assume that no Photons will be produced or anihilated on average. This means we are considering the epoch after "recombination" at $z = 1100$ (this number will be justified later). This assumption leads to

$$\underbrace{n^{(1)}_{\nu_1}}_{\text{known}} d\nu_1\, dV_1 = \underbrace{n^{(0)}_{\nu_0}}_{\text{sought}} d\nu_0\, dV_0 \qquad (8.36)$$

Here we have in mind the process

$$\boxed{\begin{matrix} n^{(1)} \\ dV_1 \\ \nu_1 \end{matrix}} \rightarrow \boxed{\begin{matrix} n^{(0)} \\ dV_0 \\ \nu_0 \end{matrix}} \qquad (8.37)$$

$n^{(1,0)}$ = Spectral distribution at $t = t^{(1,0)}$

$dV_{(1,0)}$ = co-moving volume at $t = t^{(1,0)}$

$\nu_{(1,0)}$ = Frequency at $t = t^{(1,0)}$ measured by co-moving observers

We have

$$dV_1 = \left(\frac{a_1}{a_0}\right)^3 dV_0 \qquad (8.38)$$

$$\nu_1 = \frac{a_0}{a_1} \nu_0 \qquad (8.39)$$

and therefore from (8.36)

$$n^{(0)}_{\nu_0} = n^{(1)}_{\nu_1} \frac{d\nu_1}{d\nu_0} \frac{dV_1}{dV_0}$$

$$= n^{(1)}_{\nu_1} \left(\frac{a_1}{a_0}\right)^2 \qquad (8.40)$$

$$\leadsto \; n^{(0)}_{\nu_0} = \left(\frac{a_1}{a_0}\right)^2 \frac{8\pi}{c^3} \frac{\nu_1^2}{\exp\left(\frac{h\nu_1}{kT_1}\right)-1}$$

$$= \frac{8\pi}{c^3} \frac{\nu_0^2}{\exp\left(\frac{h\nu_0}{kT_0}\right)-1} \qquad (8.41)$$

if T_0 is defined by $\nu_0/T_0 = \nu_1/T_1$ or

$$T_0 := T_1 \frac{a_1}{a_0} \qquad (8.42)$$

Hence we have

$$T_0 a_0 = T_1 a_2 \qquad (8.43)$$

or

$$T_0 = \frac{T_1}{1+z} \qquad (8.44)$$

(8.41) shows that cosmic expansion preserves the Planckian nature of the spectral distribution but changes its temperature according to (8.42).

Energy and Entropy of Black-Body radiation.

Generally we have from 1. Law of Thermodynamics

$$dE = \underbrace{T\,dS}_{\text{Heat}} - \underbrace{p\,dV}_{\text{Work}} \tag{8.45}$$

$$\Rightarrow \quad dS = \frac{1}{T}\left(dE + p\,dV\right) \tag{8.46}$$

For radiation

$$p = \frac{1}{3} u \tag{8.47}$$

where u = energy density. Note that this follows immediately from tracelessness of energy-momentum tensor

$$T^{\alpha\beta} = \frac{\varepsilon_0}{2}\left(-F^{\alpha\lambda} F^{\beta}{}_{\lambda} + \frac{1}{4} g^{\alpha\beta} F^{\lambda\sigma} F_{\lambda\sigma}\right) \tag{8.48}$$

$$g_{\alpha\beta} T^{\alpha\beta} = T^{\alpha}{}_{\alpha} = 0 \tag{8.49}$$

Now, $u = u(T)$ (8.50)

and $E = E(V,T) = V\,u(T)$ (8.51)

hence

$$dS = \frac{1}{T}\left(u\,dV + V\frac{du}{dT}dT + \frac{1}{3}u\,dV\right)$$

$$= \frac{1}{T}\left(\frac{4}{3}u\,dV + V\frac{du}{dT}dT\right) \quad (8.52)$$

This must be an exact 1-form (a "total differential"); therefore

$$\left(\frac{\partial S}{\partial V}\right)_T = \frac{4}{3}\frac{u}{T} \quad (8.53a)$$

$$\left(\frac{\partial S}{\partial T}\right)_V = \frac{V}{T}\frac{du}{dT} \quad (8.53b)$$

Integrability condition

$$\frac{\partial^2 S}{\partial V \partial T} = \frac{\partial^2 S}{\partial T \partial V} \iff \quad (8.54)$$

$$\frac{4}{3}\frac{du}{dT}\frac{1}{T} - \frac{4}{3}\frac{u}{T^2} = \frac{1}{T}\frac{du}{dT}$$

$$\iff \frac{1}{T}\frac{du}{dT} = 4\frac{u}{T^2}$$

$$\iff \frac{du}{u} = 4\frac{dT}{T} \quad (8.55)$$

$$\Leftrightarrow \quad u = a_B T^4 \qquad (8.56)$$

where a_B is actually given by

$$a_B = \frac{\pi^2 k_B^4}{15 \hbar^3 c^3} = 7.57 \cdot 10^{-16} \frac{J}{m^3 K^4} \qquad (8.57)$$

If we insert this into (8.52) we get an expression for $S(V,T)$:

$$dS = \frac{1}{T}\left(\frac{4}{3} a_B T^4 dV + V 4 a_B T^3 dT\right)$$

$$= \frac{4}{3} a_B T^3 dV + 4 a_B V T^2 dT$$

$$= d\left(\frac{4}{3} a_B T^3 V\right) \qquad (8.58)$$

$$\Rightarrow S = \frac{4}{3} a_B T^3 V + k \qquad (8.59a)$$

$k = \text{const} = 0$ (Nernst's Thm. 3. Law of Th.) (8.59b)

We have seen in (8.43) that $a \cdot T = \text{const}$ during cosmological evolution. Hence we get from (8.59a/b) and with

$$\sigma := S/V = \text{Entropy density} \qquad (8.60)$$

$$\sigma \cdot a^3 = \text{const} \qquad (8.61)$$

Since a comoving volume V satisfies

$$V_c \sim a^3 \qquad (8.62)$$

get
$$\sigma \cdot V_c = \text{const.} \qquad (8.63)$$

$\Leftrightarrow$ The entropy in a comoving volume stays constant.

If ϱ is the density of another conserved quantity, like baryon number, we have

$$\varrho \, a^3 = \text{const} \qquad (8.64)$$

$$\Rightarrow \quad \frac{\sigma}{\varrho} = \frac{\text{Entropy}}{\text{Baryon}} = \text{const.} \qquad (8.65)$$

$\Leftrightarrow$ The radiation entropy per baryon stays constant.

Also note that for the radiation pressure we have

$$p = \frac{1}{3} u = \frac{a_B}{3} T^4 \sim a^{-4} \qquad (8.66)$$

and $V_c \sim a^3$

$$\Rightarrow \quad p \cdot V_c^{\frac{4}{3}} \sim \text{const} \tag{8.67}$$

$\Rightarrow$ adiabatic index $\frac{4}{3}$

"The radiation entropy per unit mass of "dust" is, in fact, gigantic. For $T = 2.725$ K have

$$\sigma = \frac{4}{3} a_B T^3 = 2 \cdot 10^{-14} \frac{J}{m^3 K} \tag{8.68}$$

The mass density is

$$\rho_m = \frac{3 H_0^2}{8\pi G} \Omega_m \tag{8.69}$$

with $H_0 = h_0 \, 100 \, \frac{km \cdot s^{-1}}{Mpc}$

$$= h_0 \, 3.26 \, 10^{-18} \, s^{-1} \tag{8.70}$$

$(1 \, Mpc = 3.086 \cdot 10^{19} \, km)$

$$\Rightarrow \rho_m = h_0^2 \, \Omega_m \cdot 1.8 \times 10^{-26} \frac{kg}{m^3} \tag{8.71}$$

$$\Rightarrow \frac{\sigma}{\rho_m} = \frac{1.1}{\Omega_m \cdot h_0^2} \cdot 10^{12} \frac{J}{K \, kg} \tag{8.71}$$

$$= 8.3 \cdot 10^{12} \frac{J}{K \, kg} \tag{8.72}$$

for $\Omega_m = 0.27$, $h_0 = 0.7$

In comparison, the specific entropy of water at 300 K is

$$\sigma_{water}\,(300\,K) = 4.37 \cdot 10^{2}\ \frac{J}{K \cdot Kg} \qquad (8.73)$$

One can estimate that the number of CMB - photons is at least 10^4 times larger than all photons ever emitted by stars (Harrison, p. 347) and that the entropy of the CMB radiation is accordingly larger than that of other radiation by that factor. There are approximately one nucleon per m^3 in the universe, but $4 \cdot 10^8$ photons of the CMB. This ratio had been approximately constant after recombination, hence there must have been an enormous entropy production in the earlier stages of cosmic evolution. One can say that almost all entropy of the universe is stored in the 3 K CMB background (but careful: black holes are the most effective entropy storage).

In contrast to entropy, energy shows quite the opposite behaviour.
At a temperature of

$$T_{CMB} = 2.725 \text{ K} \tag{8.74}$$

We have

$$u = a_B T_{CMB}^4 = 4.2 \cdot 10^{-14} \frac{J}{m^3} \tag{8.75}$$

The energy-density of matter is given by (8.71)

$$\begin{aligned} \rho_m^0 c^2 &= h_0^2 \Omega_m \cdot 1.6 \cdot 10^{-9} \frac{J}{m^3} \\ &= 2.14 \cdot 10^{-10} \; J \cdot m^{-3} \end{aligned} \tag{8.76}$$

hence

$$\frac{\rho_m^{(0)} c^2}{u^{(0)}} = 5096 \tag{8.77}$$

where the index "0" denotes the current value (time t_0).

We have from (6.42-43)

$$c^2 \rho_{rad} = u = (1+z)^4 u^{(0)} \tag{8.78}$$

$$c^2 \rho_m = (1+z)^3 c^2 \rho_m^{(0)} \tag{8.79}$$

hence

$$\frac{c^2 \rho_m}{u} = \frac{1}{1+z} \frac{c^2 \rho_m^{(0)}}{u^{(0)}}$$

$$= \frac{5096}{1+z} \tag{8.80}$$

$\Rightarrow$ Radiation energy starts to dominate matter energy (dust) at $z \gtrsim 5100$.