Lecture 9: Horizons

In relativistic cosmology there are various notions of horizons. The most imortant ones are the so-called <u>particle horizon</u> and <u>event horizon</u>.

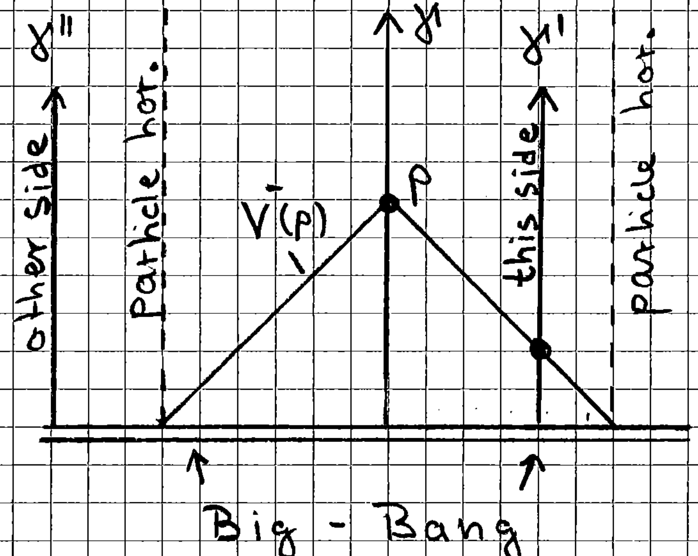
Particle horizon:

Separates the set of worldlines of "cosmological particles" (i.e. the integral lines of the cosmological vector-field $u = \partial/\partial t$) in two subsets, those on <u>this side</u> and those on the <u>other side</u> of the PH. Let γ be the integral curve of the observer and $p \in \gamma$ an event on γ (the "now"). We denote the backward lightcone at p by $V^-(p)$. Then another integral curve γ' of u (i.e. another galaxy) is called <u>on the other side</u> of the particle horizon relative to $p \in \gamma$ iff

$$\gamma' \cap V^-(p) = \phi \tag{9.1}$$

and on this side iff $\gamma' \cap V^-(p) \neq \phi$

Note that whether or not γ' is intersected by $V^-(p)$ depends on the reference event p. Any event $p \in M$ therefore partitions the set of all integral curves γ' of u into two subsets, according to whether (9.1) holds or not.



(9.2)

The figure (9.2) shows the typical situation in the presence of a "Big-Bang" singularity, i.e. if $a \searrow 0$ for some time t. Since

$$1 + z = \frac{a_0}{a(t)} \qquad (9.3)$$

$a(t) \searrow 0$ corresponds to $z \nearrow \infty$.

Another way to formulate this is to use a conformally static representation of the FLRW metric using conformal time η, defined by

$$d\eta = \frac{dt}{a(t)} \tag{9.4}$$

Then

$$g = c\,dt \otimes c\,dt - a^2(t)\,\hat{g}$$

$$= a^2(\eta) \underbrace{\left[c\,d\eta \otimes c\,d\eta - \hat{g} \right]}_{\text{Static.}} \tag{9.5}$$

where

$$\hat{g} = d\chi \otimes d\chi + \Sigma_k^2(\chi)\, d^2\Omega \tag{9.6a}$$

$$= \frac{dr^2}{1-kr^2} + r^2\, d^2\Omega . \tag{9.6b}$$

For radial lightlike geodesics we have

$$c\,d\eta = \pm\, d\chi \tag{9.7a}$$

$$= \pm \frac{dr}{\sqrt{1-kr^2}} \tag{9.7b}$$

so that η is a direct measure for how far in space light travels.

Now

$$d\eta = \frac{dt}{a(t)} = \frac{da}{\dot{a}\,a} = \frac{da}{Ha^2} \qquad (9.8)$$

and $(1+z) = \frac{a_0}{a}$

$$\Rightarrow \quad dz = -\frac{a_0}{a^2}\,da \qquad (9.9)$$

$$\Rightarrow \quad d\eta = -\frac{dz}{a_0 H(z)} \qquad (9.10)$$

Using (6.49) and (7.10), i.e.

$$H(z) = H_0 \left\{ (1+z)^4 \Omega_{rad} + (1+z)^3 \Omega_{dust} + (1+z)^2 \Omega_k + \Omega_\Lambda \right\}^{1/2}$$

we get for the amount of conformal time between now ($t = t_0$, $\eta = \eta_0$, $z = 0$) and the Big-Bang ($z = \infty$)

$$\int_{\eta_{BB}}^{\eta_0} d\eta = (\eta_0 - \eta_{BB})$$

$$\frac{1}{a_0 H_0} \int_0^\infty \frac{dz}{\sqrt{(1+z)^4 \Omega_r + (1+z)^3 \Omega_d + (1+z)^2 \Omega_k + \Omega_\Lambda}} \qquad (9.11)$$

which is finite – i.e. converges at upper limit $z = \infty$ – iff Ω_d or $\Omega_r > 0$.

This implies that if $\Omega_{dust} > 0$ and / or $\Omega_{rad} > 0$ the backward light cone only covers a finite diameter region of space in the $\hat{g}$ - metric at $z = \infty$ $(a = 0)$.
Unless the space is compact, e.g. S^3, and the distance $c(\eta_0 - \eta_{BB})$ is larger than the circumference of S^3 in the metric $\hat{g}$, there will be a particle horizon of radius

$$\hat{R}_P = c(\eta_0 - \eta_{BB}) \tag{9.12}$$

in $\hat{g}$ metric.

Since all realistic cosmologies satisfy this condition of $\Omega_m > 0$, $\Omega_{rad} > 0$ particle horizons are a generic feature of FLRW models. For an explicit example with compact space, see Problem-Sheet 5.

Event horizon

Separates relative to a given world-line γ the set of events, i.e space-time itself, into two sets: Those events of which γ can know are on this side of the event-horizon, and those event of which γ never knows (i.e. never receives a causal signal) are on the other. Note that an event horizon is defined relative to the whole timelike curve γ; it does not refer to a specific event $p \in \gamma$, unlike the particle horizon.

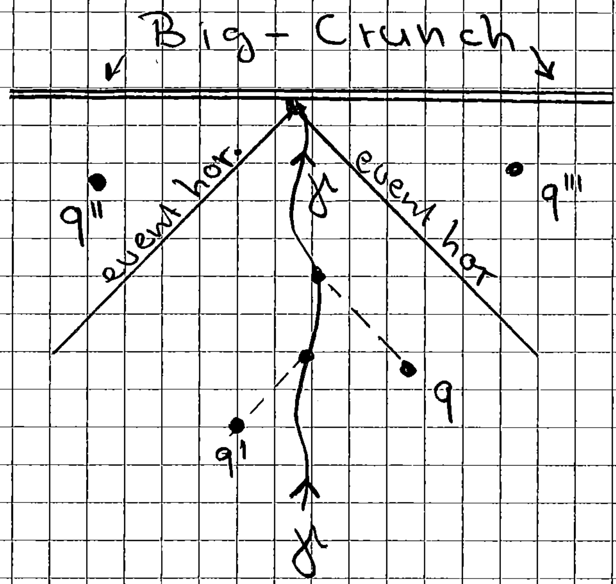
Let $$V^-(\gamma) = \bigcup_{p \in \gamma} V^-(p) \qquad (9.13)$$

be the set of events of which the observer can recieve a light signal throughout his/her history. Then the event $p \in M$ is on this side of the event horizon if

$$p \in V^-(\gamma) \qquad (9.14)$$

and on the other if

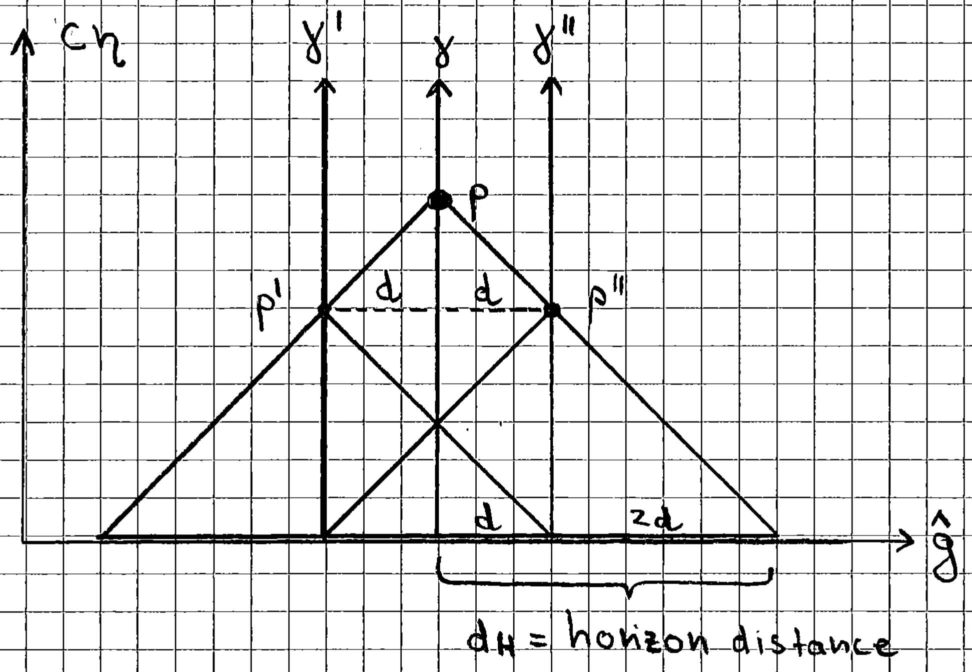
$$p \notin V^-(\gamma) \qquad (9.15)$$



(9.16)

Observer "sees" q' and q, but never q'' and q'''. The event horizon is the backward lightcone at the event at which γ ends, e.g. because of a "Big-Crunch" singularity.

An important consequence of particle horizons is that an observer may simultaneously "see" cosmological particles which do not yet have had any causal contact, i.e. which up to the time at which the observer sees them, cannot see each other. This we show in the following Figure, which again maps the FLRW geometry to a conformally equivalent one.

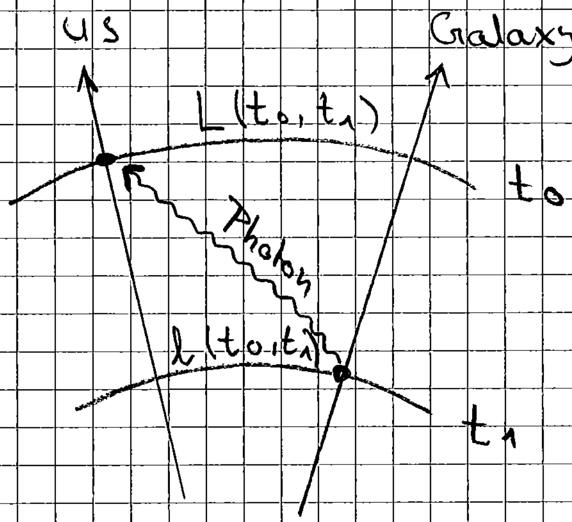


This figure shows that "galaxies" γ' and γ'' seen at event $p \in \gamma$ by observer γ in diametrically opposite directions (at $p' \in \gamma'$ and $p'' \in \gamma''$) just cannot see each other if the (conformal) $\hat{g}$-distance of γ' and γ'' to γ (at the moment of emission, but that does not matter in $\hat{g}$-metric which is time independent) is a third of particle-horizon distance d_H (in $\hat{g}$):

$$d = \frac{1}{3} d_H. \qquad (9.17)$$

How far can we see in the universe?

The two geodesic distances that we already introduced in (7.26) are



(9.18)

where we now call – for simplicity

$L(t_0, t_1)$ = geodesic distance to us time t_0 of reception (9.19)

$l(t_0, t_1)$ = geodesic distance to us at time of emission (9.20)

Note that we now explicitly denote both times, t_0 = time of reception and t_1 = time of emission, as arguments. Their analytic expressions are as follows:

$$L(t_0, t_1) = c\, a(t_0) \int_{t_1}^{t_0} \frac{dt}{a(t)} \qquad (9.21)$$

$$l(t_0, t_1) = c\, a(t_1) \int_{t_1}^{t_0} \frac{dt}{a(t)} \qquad (9.22)$$

or in terms of z, again with the standard substitutions

$$\frac{dt}{a} = \frac{da}{\dot{a}\, a} = \frac{da}{H a^2}$$

$$1+z = \frac{a_0}{a} \rightsquigarrow dz = -\frac{a_0}{a^2}\, da$$

$$\Rightarrow \frac{dt}{a(t)} = -\frac{1}{a_0} \frac{dz}{H(z)}$$

$$L(z) = c \int_0^z \frac{dz'}{H(z')} \qquad (9.23)$$

$$l(z) = \frac{c}{1+z} \int_0^z \frac{dz'}{H(z')} \qquad (9.24)$$

We apply this to "Big-Bang" models in which $a \to 0$ for $t \to t_*$, where t_* is often $= 0$ or $-\infty$. Note that $a \to 0$ corresponds to $z \to \infty$, as already used.

Note that L in (9.21), considered as function of 2nd argument t_1 is monotonically decreasing:

$$t_1 \mapsto L(t_0, t_1) = c \int_{t_1}^{t_0} \frac{dt}{a(t)} \tag{9.25}$$

monotonically decreasing
for $t_1 \in [t_*, t_0]$

The supremum of this function is just the particle horizon in the physical metric $a(t_0)\,\hat{g}$:

$$L_p(t_0) := \lim_{t_1 \to t_*} L(t_0, t_1) \tag{9.26}$$

t_*: t-value for $a(t_*) = 0$.

$L_p(t_0)$ is the maximal geodesic distance of other Galaxies <u>now</u> that we receive light from.

In contrast, consider l in (9.22) as function of 2nd argument. We have

$$l(t_0, t_*) = \underbrace{c\, a(t_*)}_{=0} \int_{t_*}^{t_0} \frac{dt}{a(t)} = 0 \tag{9.27}$$

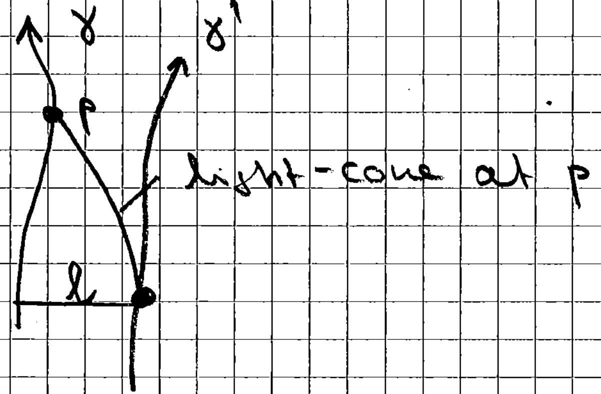
and finally

$$\ell(t_0, t_0) = c\, a(t_0) \underbrace{\int_{t_0}^{t_0} \frac{dt}{a(t)}}_{=0} = 0 \qquad (9.28)$$

Hence

$$t_1 \mapsto \ell(t_0, t_1) = c\, a(t_1) \int_{t_1}^{t_0} \frac{dt}{a(t)} \qquad (9.29)$$

for $t_1 \in [t_*, t_0]$ has zeros at end points of intervall and is positive inbetween. This function characterises the backward light cone



We have

$$\frac{\partial \ell(t_0, t_1)}{\partial t_1} = c\, \dot{a}(t_1) \int_{t_1}^{t_0} \frac{dt}{a(t)}$$

$$- c\, a(t_1) \frac{1}{a(t_1)}$$

$$= H(t_1)\, \ell(t_0, t_1) - c \qquad (9.30)$$

Differentiating once more using

$$\dot{H} = \left(\frac{\dot{a}}{a}\right)^{\cdot} = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}$$

$$= -\left(\frac{\dot{a}}{a}\right)^2 \left[1 - \frac{\ddot{a}\,a}{\dot{a}^2}\right]$$

$$= -H^2(1+q) \qquad (9.31)$$

where

$$q(t) = -\frac{\ddot{a}(t)\,a(t)}{\dot{a}^2(t)} \qquad (9.32)$$

is the <u>deceleration parameter</u>.

Hence

$$\frac{\partial^2 l(t_0,t_1)}{\partial t_1^2} = -(1+q(t_1))\,H^2(t_1)\,l(t_0,t_1) + H(t_1)\left(H(t_1)\,l(t_0,t_1) - c\right)$$

$$= -c\,H(t_1)\left[1 + q(t_1)\frac{l(t_0,t_1)\,H(t_1)}{c}\right] \qquad (9.33)$$

At $t_1 = t_{max}$ where (9.30) is zero, i.e. where

$$l(t_0,t_{max}) = \frac{c}{H(t_{max})} \qquad (9.34)$$

We have

$$\left.\frac{\partial^2 l(t_0, t_1)}{\partial^2 t_1}\right|_{t_1 = t_{max}} =$$

$$- c H(t_{max}) \left(1 + q(t_{max}) \right) \qquad (9.35)$$

which is a maximum for

$$q(t_{max}) > -1 \qquad (9.36)$$

Equation (9.34) defines t_{max} as function of t_0 implicitly; we denote it by

$$t_{max} = f(t_0) \qquad (9.37)$$

The spherical intersections of the past lightcone with the hypersurfaces of constant time reach their maximal metric diameter for $t = t_{max}$. For $t < t_{max}$ their diameters decrease again.

Def. The 2-sphere with geodesic radius $l(t_0, t_{max})$ and centre $\chi = 0$ within the hypersurface $t = t_{max} = f(t_0)$ is called the <u>Hubble-Horizon</u> for the event $(t = t_0, \chi = 0)$. The corresponding geodesic distance in the $t = t_{max}$ hypersurface $t = t_{max}$, $l(t_0, t_{max})$, is called the <u>Hubble Radius</u>. for the event $(t = t_0, \chi = 0)$

Relative Velocities in FLRW Cosmologies

The simultaneous geodesic distance between two cosmological "particles" ("objects") like "us" at $\chi = 0$ and a "Galaxy" at χ, is given by $d(t) = a(t)\,\chi$. Hence the simultaneous velocity with which the Galaxy recedes from us is

$$V(\chi, t) = \dot{d}(t) = \chi\, \dot{a}(t)$$

$$= H(t)\, d(t) \qquad (9.38)$$

Note that $V(\chi, t)$ is the relative simultaneous velocity (proper-time rate of change of simultaneous geodesic distance) between the cosmological "particles" at time t which are a radial coordinate χ apart.

From (9.34) we thus see that that an object on our Hubble horizon

$$l_H(t_0) := l(t_0, t_{max}) \tag{9.39}$$

had a relative simultaneous velocity to us at the time it emitted its light, i.e. at $t = t_{max}$, of

$$\begin{aligned} V(l_H(t_0), t_{max}) &= H(t_{max})\, l(t_0, t_{max}) \\ &= c \end{aligned} \tag{9.40}$$

In that sense the relative "object velocity" between "us" and Galaxies on our Hubble horizon was c at the time t_{max} the Galaxy emitted its light.

Different from that is the velocity with which the Hubble Horizon itself expands, i.e. the time-rate of change of $l(t_0, t_{max})$. Here "time-rate of change" may refer to the time of reception, t_0, or the time of emission, t_{max}.

For the latter we set

$$V_H^{(1)}(t_{max}) := \frac{d\, l(t_0, t_{max})}{d t_{max}}$$

$$\overset{(9.34)}{=} \frac{d}{dt_{max}} \left(\frac{c}{H(t_{max})} \right)$$

$$\overset{(9.31)}{=} c\,\left(1 + q(t_{max})\right) \qquad (9.41)$$

If the Universe is expanding in an accelerated manner at the time of emission, i.e. $q(t_{max}) < 0$, then $V_H^{(1)}(t_{max}) < c$ and the Objects ("Galaxies") overtake the Hubble Horizon. That is, relative to the Hubble Horizon the Galaxies cross the Horizon from the inside to the outside.

Conversely, if the universe expanded in a decelerating fashion at the time of emission, i.e. $q(t_{max}) > 0$, then $V_H^{(1)}(t_{max}) > c$ and the Hubble horizon expands faster than the Galaxies on it. In other words: The Galaxies cross the Horizon from the outside to the inside.

We can also differentiate (9.34) with respect to t_0:

$$V_H^{(0)}(t_0) := \frac{d\,l(t_0, t_{max} = f(t_0))}{dt_0}$$

$$= \frac{d}{dt_0}\left(\frac{c}{H(f(t_0))}\right)$$

$$= V_H^{(1)}(t_{max})\,\dot{f}(t_0) \tag{9.42}$$

The derivative $\dot{f}$ can be calculated from the equation implicitly defining f:

$$l(t_0, f(t_0)) \equiv \frac{c}{H(f(t_0))} \tag{9.43}$$

where the $\equiv$ sign is meant to indicate that this equation holds for all t_0. Hence we may differentiate this equation with respect to t_0. From the definition (9.22) of $l(t_0, t_1)$,

$$l(t_0, t_1) = c\, a(t_1) \int_{t_1}^{t_0} \frac{dt}{a(t)} \qquad (9.22)$$

We get

$$\frac{\partial l(t_0, t_1)}{\partial t_0} = c\, \frac{a(t_1)}{a(t_0)} \qquad (9.44)$$

and from the definition of t_{max}

$$\left.\frac{\partial l(t_0, t_1)}{\partial t_1}\right|_{t_1 = t_{max} = f(t_0)} = 0 . \qquad (9.45)$$

Hence, taking d/dt_0 of (9.43) gives on the left-hand side

$$\frac{d}{dt_0}\, l(t_0, f(t_0)) = c\, \frac{a(f(t_0))}{a(t_0)} \qquad (9.46)$$

and on the right-hand side, using (9.31):

$$\frac{d}{dt_0}\left(\frac{c}{H(f(t_0))}\right) = c\,(1 + q(f(t_0)))\, \dot{f}(t_0) \qquad (9.47)$$

Equating (9.46) and (9.47) allows to solve for $\dot{f}(t_0)$:

$$\dot{f}(t_0) = \frac{a(t_{max}) / a(t_0)}{1 + q(t_{max})} \tag{9.48}$$

where we set again $t_{max} = f(t_0)$.

Replacing the two terms $V_H^{(1)}$ and $\dot{f}$ on the right-hand side by (9.41) and (9.48) we get for $V_H^{(0)}$:

$$V_H^{(0)}(t_0) = c\,(1 + q(t_{max}))$$

$$\times \quad \frac{a(t_{max}) / a(t_0)}{1 + q(t_{max})}$$

$$= c \cdot \frac{a(t_{max})}{a(t_0)} \tag{9.49a}$$

$$= \frac{c}{1+z} \tag{9.49b}$$

where z here is the redsift factor at t_{max}

Next to the "expansion velocity of the Hubble-Horizon we can also calculate that of the particle Horizon

From (9.21)

$$L(t_0, t_1) = c\, a(t_0) \int_{t_1}^{t_0} \frac{dt}{a(t)} \qquad (9.21)$$

and $L_p(t_0) = L(t_0, t_*)$

we get

$$V_p^{(0)}(t_0) = \frac{d}{dt_0} L(t_0, t_1) \Big|_{t_1 = t_*}$$

$$= H(t_0)\, L_p(t_0) + c \qquad (9.50)$$

At time t_0 of observation the objects at the particle horizon recede from us at a velocity given by (9.38)

$$V(L_p(t_0), t_0) = H(t_0)\, L_p(t_0) \qquad (9.51)$$

Comparison with (9.50) shows that the particle Horizon sweeps over the "particles" with a simultaneous relative velocity of c.

Examples

We consider models of constant deceleration parameter

$$q := -\frac{\ddot{a}\, a^2}{\dot{a}^2} = \text{const} \tag{9.52}$$

$$\Leftrightarrow \quad \frac{\ddot{a}}{\dot{a}} = -q\,\frac{\dot{a}}{a}$$

$$\Leftrightarrow \quad \ln\left(\dot{a}\cdot a^{q}\right) = k = \text{const}$$

$$\Leftrightarrow \quad \dot{a}\, a^{q} = e^{k}$$

$$\Leftrightarrow \quad a^{q}\, da = e^{k}\, dt$$

$$\Leftrightarrow \quad a(t) = \begin{cases} k\, t^{\frac{1}{1+q}} & \text{for } q > -1 \\ a_0 e^{kt} & \text{for } q = -1 \end{cases} \tag{9.53}$$

where k is a positive constant.
We restrict to models with particle horizon. For those

$$\lim_{t_1 \to t_*} \int_{t_1} \frac{dt}{a(t)} < \infty \tag{9.54}$$

where $a(t_*) = 0$ ($t_* = 0$ for $q > -1$
$t_* = -\infty$ for $q = -1$).

Hence for this integral to converge at lower limit $t_1 = 0$ the integrand $t^{-\frac{1}{1+q}}$ must have an exponent > -1;

$$-\frac{1}{1+q} > -1$$

$$\Leftrightarrow \quad q > 0 \qquad (q > -1). \tag{9.55}$$

The functions $L(t_0, t_1)$ in (9.21) and $l(t_0, t_1)$ can now be calculated. In fact, it is sufficient to calculate one of them; the other is obtained by exchanging t_0 and t_1 and multiplication with (-1). The straightforward results are

$$L(t_0, t_1) = c\, t_0 \, \frac{1+q}{q} \left(1 - \left(\frac{t_1}{t_0}\right)^{\frac{q}{1+q}}\right) \tag{9.56}$$

$$l(t_0, t_1) = c\, t_1 \, \frac{1+q}{q} \left(\left(\frac{t_0}{t_1}\right)^{\frac{q}{1+q}} - 1\right) \tag{9.57}$$

We also have

$$1 + z = \frac{a(t_0)}{a(t_1)} = \left(\frac{t_0}{t_1}\right)^{\frac{1}{1+q}} \tag{9.58}$$

$$\Rightarrow \quad H(t) = \frac{1}{(1+q)t} = H_0 \, \frac{t_0}{t} \tag{9.59}$$

$$H(z) = H_0 \, (1+z)^{(1+q)} \tag{9.60}$$

From (9.57) we can calculate t_{max} according to (9.45). For this we first rewrite:

$$\ell(t_0, t_1) = c \frac{1+q}{q} t_1 \left[\left(\frac{t_0}{t_1}\right)^{\frac{q}{1+q}} - 1 \right]$$

$$= c \frac{1+q}{q} \left[t_0^{\frac{q}{1+q}} t_1^{\frac{1}{1+q}} - t_1 \right] \qquad (9.61)$$

$$\Rightarrow \frac{\partial}{\partial t_1} \ell(t_0, t_1) =$$

$$c \frac{1+q}{q} \left[t_0^{\frac{q}{1+q}} \frac{1}{1+q} t_1^{-\frac{q}{1+q}} - 1 \right]$$

$$= \frac{c}{q} \left[\left(\frac{t_0}{t_1}\right)^{\frac{q}{1+q}} - (1+q) \right] \qquad (9.62)$$

Hence $\partial \ell(t_0, t_1) / \partial t_1 = 0$ iff

$$t_1 = t_{max} = t_0 (1+q)^{-\left(\frac{1+q}{q}\right)}$$

$$= H_0^{-1} (1+q)^{-(1+2q)/q} \qquad (9.63)$$

Where we used (9.59), i.e. that $H_0 = 1/(1+q) t_0$ in the last step. Using this $t_1 = t_{max}$ in (9.61) we get

$$\ell_H(t_0) = \ell(t_0, t_{max}) = \frac{c}{H_0} (1+q)^{-(1+q)/q} \qquad (9.64)$$

The particle horizon is

$$L_P(t_0) = L(t_0, t_1 = 0)$$

$$= c\,t_0 \,\frac{1+q}{q}$$

$$= \frac{c}{H_0}\,\frac{1}{q} \qquad (9.64)$$

using again (9.59).

We can use (9.58 - 60) to express (9.56, 57), i.e. l and L, as function of z. For that we replace t_0/t_1 by $(1+z)^{(1+q)}$ and in the prefactors $t_1(1+q)$ by $H_1^{-1} = H_0^{-1}(1+z)^{-(1+q)}$ and $t_0(1+q)$ by H_0^{-1}. Then we get

$$L(z) = \frac{c}{H_0 q}\left[1 - (1+z)^{-q}\right] \qquad (9.65)$$

$$l(z) = \frac{c}{H_0 q}\left[1 - (1+z)^{-q}\right] \cdot \frac{1}{(1+z)} \qquad (9.66)$$

The redshift with which the observer sees the "Galaxy" at the Hubble-Horizon, i.e at largest distance l is either obtained by directly calculating the extrema of (9.66)

$$l'(z) = \frac{c}{H_0 q}\left[-(1+z)^{-2} + (q+1)(1+z)^{-q-2}\right]$$

$l'(z) = 0 \iff z = z_H^{(1)}$, where

$$z_H^{(1)} = (1+q)^{1/q} - 1 \qquad (9.67)$$

This is the cosmological redshift at which the observer sees the "Galaxy" which at the emission time t_1 had the simultaneous distance $l(t_0, t_1)$ equal to $c/H(t_1)$. Note that this means that $t_1 = t_{max}$ according to (9.34).

Similarly we can calculate the redshift of the galaxy which at the time t_0 of observation has the Hubble distance H_0/c. This immediately follows from (9.65) if we there set $L(z) = c/H_0$:

$$z_H^{(0)} = (1-q)^{-1/q} - 1 \qquad (9.68)$$

Note the (subtle) difference between (9.67) and (9.68). For example, for a matter-dominated universe $\Omega_{rad} = \Omega_\Lambda = \Omega_K = 0$, $\Omega_{dust} = 1$ we have (compare Problem 5, Sheet 4):

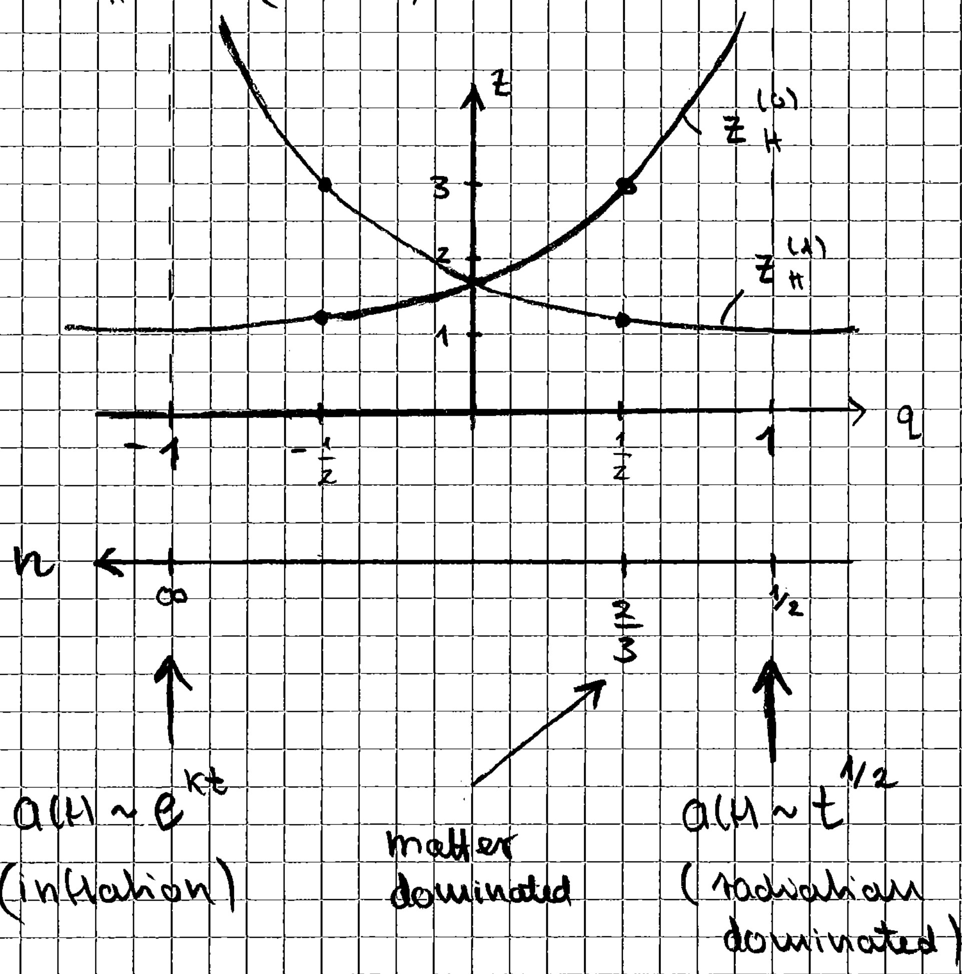
$$a(t) = A t^{2/3}$$

$$\dot{a} = \frac{2}{3} A t^{-1/3}, \quad \ddot{a} = -\frac{2}{9} A t^{-4/3}$$

$$q = -\frac{\ddot{a} a}{\dot{a}^2} = \frac{2/9}{(2/3)^2} = \frac{1}{2} \qquad (9.69)$$

$$\Rightarrow z_H^{(1)} = \left(1 + \frac{1}{2}\right)^2 - 1 = \frac{5}{4} \qquad (9.70)$$

$$z_H^{(0)} = \left(1 - \frac{1}{2}\right)^{-1/2} - 1 = 3 \qquad (9.71)$$



(9.72)

Causally disconnected regions in the surface of last scattering / recombination

Let t_{re} denote the time of "recombination", i.e. the time in cosmic evolution when the temperature fell below the ionisation energy of H (13.6 eV). This roughly happened at

$$z_{re} \cong 1100 \tag{9.73}$$

(This will be discussed on Sheet 5)

We assume a matter dominated flat universe:

$$\left.\begin{aligned} \Omega_{dm} &= 1 \\ \text{all other } \Omega's &= 0 \end{aligned}\right\} \tag{9.74}$$

Then

$$\left.\begin{aligned} a(t) &= A\, t^{2/3} \\ \text{and} \quad H(t) &= \frac{2}{3} t^{-1} \\ q(t) &= q = \frac{1}{2} \end{aligned}\right\} \tag{9.75}$$

The particle-horizon at time $t = t_{re}$ follows from (9.56):

$$R_{re} = L_p(t_0 = t_{re}, 0) = 3 c\, t_{re} \tag{9.76}$$

The geodesic distance at observation time t_0 to the surface (in the sky) of last scattering is, again from (9.56),

$$L(t_0, t_{re}) = 3ct_0\left[1-\left(\frac{t_{re}}{t_0}\right)^{1/3}\right]. \quad (9.77)$$

The visual angle under which we see the particle horizon on the surface of last scattering is

$$\Delta\varphi(D=R_{re}, z_{re}) = \frac{R_{re}}{d_A(z_{re})} \quad (9.78)$$

(due to def. of d_A)

$$\text{But } d_A = d_a^{(1)} \quad (\text{for } k=0)$$

$$= d_a^{(0)}/(1+z)\Big|_{z=z_{re}}$$

$$= L(t_0, t_{re})/(1+z_{re}) \quad (9.79)$$

Hence

$$\Delta\varphi(D=R_{re}, z_{re}) = (1+z_{re})\frac{R_{re}}{L(t_0, t_{re})}$$

$$= (1+z_{re})\frac{L(t_{re}, 0)}{L(t_0, t_{re})}$$

$$= (1+z_*)\frac{t_{re}}{t_0}\left[1-\left(\frac{t_{re}}{t_0}\right)^{1/3}\right]^{-1} \quad (9.80)$$

But by (9.58)

$$\frac{t_{re}}{t_0} = \left(\frac{1}{1+z_{re}}\right)^{3/2} \qquad (9.81)$$

hence

$$\Delta\varphi = \frac{1}{\sqrt{1+z_{re}}}\left[1 - \frac{1}{\sqrt{1+z_{re}}}\right]^{-1}$$

$$= \frac{1}{\sqrt{1+z_{re}} - 1} \qquad (9.82)$$

With $z_{re} \cong 1100$

$$\Rightarrow \Delta\varphi \cong 3.1 \times 10^{-2}$$

$$= 1.8^{\circ} \qquad (9.83)$$

$\Rightarrow$ Two points in the CMB where angular seperation is larger than 1.8° = 4 x full moon, are causally disconnected if we assume $a(t) \sim t^{2/3}$ (matter dominated) universe

See Sheet 5 for $a(t) \sim t^{1/2}$ (radiation dominated) case.