Exercises for the lecture on

Special Topics in GR & Relativistic Cosmology

by DOMENICO GIULINI

Sheet 1

Problem 1

The Poisson equation for the Newtonian gravitational potential ϕ reads

$$\Delta\phi = 4\pi G\rho\,, \tag{1}$$

where $\rho \geq 0$ is the matter density.

Give the most general solution for constant ρ if space is A) $\mathbb{R}^3$ and B) the round 3-sphere S^3 with radius R.

Problem 2

Consider a set of N point-masses m_a, $a = 1, \cdots, N$, under the influence of their own gravitational attraction according to Newton's law. The $3N$ equations of motion are given by

$$m_a \ddot{\vec{x}}_a(t) = \sum_{\substack{b=1 \\ b\neq a}}^{N} G m_a m_b \frac{\vec{x}_b(t) - \vec{x}_a(t)}{\|\vec{x}_b(t) - \vec{x}_a(t)\|^3}\,. \tag{2}$$

Show that

$$m_a \ddot{\vec{x}}_a(t) = -\vec{\nabla}_a V\big(\vec{x}_1(t), \cdots, \vec{x}_N(t)\big)\,, \tag{3a}$$

where $\vec{\nabla}_a := \partial/\partial\vec{x}_a$ and

$$V(\vec{x}_1, \cdots, \vec{x}_N) = -\frac{1}{2}\sum_{\substack{a,b=1 \\ a\neq b}}^{N} \frac{G m_a m_b}{\|\vec{x}_b - \vec{x}_a\|}\,. \tag{3b}$$

Show further that

$$\sum_{a=1}^{N} \vec{x}_a \cdot \vec{\nabla}_a V = -V\,. \tag{4}$$

Problem 3

This is a continuation of the pervious problem.

Seek solutions of (3) of the form (called "homothetic motions")

$$\vec{x}_a(t) = a(t)\vec{y}_a \tag{5}$$

with the same non-negative function $a(t)$ for all a and N time-independent vectors $\vec{y}_a$. Any N-tuple of vectors $(\vec{y}_1, \cdots, \vec{y}_N)$ for which a solution to (2) exists is called a *central configuration*. The aim of this and the following problems is to discuss, as complete as possible here, the restrictions the equations of motion (3) imposes onto the function $a(t)$ and upon the locations $\vec{y}_a$.

Show first that $a(t)$ must satisfy a differential equation of the form

$$\tfrac{1}{2}\dot{a}^2 + \frac{C}{a} = E \tag{6}$$

where C and E are constants. Furthermore, show that the constant C is given by

$$C := \ddot{a}a^2 = \frac{V(\vec{y}_1, \cdots, \vec{y}_N)}{\sum_{a=1}^N m_a \|\vec{y}_a\|^2} = -\kappa < 0 \tag{7}$$

and hence negative. The modulus of C is called κ.

Problem 4

Find solutions of (3) for negative, zero, and positive E.

Problem 5

Show that if $(\vec{y}_1, \cdots, \vec{y}_N)$ is a central configuration, so is $(c\vec{y}_1, \cdots, c\vec{y}_N)$, where $c \in \mathbb{R} - \{0\}$. Argue that therefore the search for central configurations may without loss of generality be be restricted to those on the $(3N-1)$-dimensional ellipsoid

$$\sum_{a=1}^N m_a \|\vec{y}_a\|^2 = 1\,. \tag{8}$$

Show further that if $(\vec{y}_1, \cdots, \vec{y}_N)$ is a central configuration, so is $(D\vec{y}_1, \cdots, D\vec{y}_N)$, where $D \in SO(3)$ is a rotation matrix.

Problem 6

Show that the condition for $(\vec{y}_1, \cdots, \vec{y}_N)$ being a central configuration is equivalent to (as in (7) we write $\kappa := -C$, so as to have $\kappa > 0$)

$$\kappa m_a \vec{y}_a + \sum_{\substack{b=1 \\ b\neq a}}^N G m_a m_b \frac{\vec{y}_b - \vec{y}_a}{\|\vec{y}_b - \vec{y}_a\|^3} = 0\,. \tag{9}$$

Show that this implies

$$\sum_{a=1}^{N} m_a \vec{y}_a = \vec{0}\,. \tag{10}$$

Show the general identity ($M = \sum_a m_a$ denotes the total mass)

$$m_a \vec{y}_a = \frac{1}{M} \sum_{\substack{b=1 \\ b \neq a}}^{N} m_a m_b (\vec{y}_a - \vec{y}_b) + \frac{m_a}{M} \sum_{b=1}^{N} m_b \vec{y}_b\,. \tag{11}$$

Use this and (10) to rewrite (9) as

$$\sum_{\substack{b=1 \\ b \neq a}}^{N} \vec{F}_{ab} = \vec{0}\,, \tag{12a}$$

where

$$\vec{F}_{ab} = m_a m_b (\vec{y}_a - \vec{y}_b) \left(\frac{\kappa}{M} - \frac{G}{r_{ab}^3} \right) \tag{12b}$$

and $r_{ab} := \|\vec{y}_a - \vec{y}_b\|$ denote the mutual distances.

Hence special central configurations are given if the N mass points can be arranged in such a way that all 2-particle distances are the same and equal to

$$r_{ab} = \left(\frac{MG}{\kappa} \right)^{1/3}\,. \tag{13}$$

Note the remarkable fact that this holds independent of whether the masses making up M are equal or vastly different. What would possible configurations for $N = 2, 3, 4$ be? What about $N = 5$?

Another interpretation of equation (9) is by seeking the stationary points of the positive real-valued function $F(\vec{y}_1, \cdots, \vec{y}_N) := -V(\vec{y}_1, \cdots, \vec{y}_N)$ with constraints (Nebenbedingungen) that $\|\vec{y}_a\| = R$ for all a. Mathematically the stationary points correspond to the lowest energy configuration of N positive charges placed on a 2-sphere of radius R. This lends some physical intuition to central configurations.