Exercises for the lecture on

Special Topics in GR & Relativistic Cosmology

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Sheet 2

Problem 1

Consider Newtonian Mechanics in which the inertial structure differs from the standard one in the following sense: Spacetime is a 4-dimensional real affine space which upon choosing a basis we identify with $\mathbb{R} \times \mathbb{R}^3$. The usual inertial structure is that in which force-free motion defines curves $\lambda \mapsto \big(t(\lambda)\,,\,\vec{x}(\lambda)\big)$ so that $\dot{t} = 1$ and $\ddot{\vec{x}} = \vec{0}$. The *modified inertial structure* is that in which a particular homothetic motion

$$\dot{\vec{x}}(t) = \frac{\dot{R}(t)}{R(t)}\vec{x}(t) \tag{1}$$

is force free, for some specified function $t \mapsto R(t)$. The corresponding *modified Newtonian equation of motion* for a point particle of mass m is then obtained by letting the force be equal to the acceleration relative to the homothetic (inertial) motion.

Show that this results in

$$\vec{F} = m\Big(\ddot{\vec{x}} - (\ddot{R}/R)\,\vec{x}\Big)\,. \tag{2}$$

Imagine you live in such a space (universe) and you release a point mass at time $t = 1$ at $\vec{x} = (r, 0, 0)$ with vanishing velocity $\dot{\vec{x}}(1) = \vec{0}$. Discuss its motion if

A) $R(t) \propto t^{2/3}$,

B) $R(t) \propto \exp(\lambda t)$ for $\lambda > 0$ and $\lambda < 0$.

Problem 2

Consider again the modified Newtonian equation of motion (2) in an exponentially expanding universe $R(t) \propto \exp(\lambda t)$, $\lambda > 0$. Specialise to the Kepler-Problem where $\vec{F} = -m\vec{\nabla}V$ an $V = V(r) = -C/r$ with $C > 0$ and $r := \|\vec{x}\|$.

Give a qualitative discussion of the motion in terms of the effective potential for the one-dimensional motion in r after using angular-momentum conservation. Discuss and explain the dependence of the radius of a bound stable circular orbit of fixed angular momentum upon the value of λ. Show that there exists a critical radius

$$r_c := \left(\frac{C}{\lambda^2}\right)^{1/3} \tag{3}$$

above which no bound circular orbit exists. Calculate the approximate value of r_c, given that $\lambda^2 = -H_0^2 q_0$ (prove that) and that the approximate values for the Hubble constant and deceleration parameter are

$$H_0 := \frac{\dot{a}}{a}\Big|_{t=t_0} \approx 70\,\mathrm{km}\cdot\mathrm{s}^{-1}\cdot(\mathrm{Mpc})^{-1}\,, \qquad q_0 := -\frac{a\ddot{a}}{\dot{a}^2}\Big|_{t=t_0} \approx -0.6\,. \tag{4}$$

Problem 3

We represent Minkowski space by $(\mathbb{R}^4, \eta)$, where with respect to standard coordinates $x^\alpha = (ct, \vec{x})$ we have $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$. We write: $\eta(x,x) = \eta_{\alpha\beta}x^\alpha x^\beta = c^2t^2 - r^2$, where $r^2 = \vec{x}\cdot\vec{x}$.

In Minkowski space we regard the "wedge-region" W given by

$$W := \{(ct, \vec{x}) \in \mathbb{R}^4 : ct > r\}\,. \tag{5}$$

In W we define the radial vector field

$$u = c\frac{x^\alpha}{\sqrt{\eta(x,x)}}\partial_\alpha\,. \tag{6}$$

Show by direct calculation that it is geodesic and parametrised by proper time (arc-length divided by c). Give a simple argument (without any calculation) as to why the geodesic nature is obvious.

Now regard (W, η, u) as a cosmological model for $\Lambda = 0$ and in the limiting case of vanishing energy-momentum tensor. We chose coordinates $(\tau, \rho, \theta, \varphi)$ in W according to

$$ct = c\tau\,\cosh(\rho)\,, \tag{7a}$$

$$\vec{x} = c\tau\,\sinh(\rho)\,\vec{n}\,, \tag{7b}$$

where

$$\vec{n} = \begin{pmatrix}\sin\theta\,\cos\varphi\\ \sin\theta\,\sin\varphi\\ \cos\theta\end{pmatrix}\,. \tag{7c}$$

Show that

$$\eta = c\,d\tau\otimes c\,d\tau - a^2(\tau)\,h \tag{8a}$$

where

$$a(\tau) = c\tau \tag{8b}$$

is the "scale factor" and

$$h = d\rho\otimes d\rho + \sinh^2(\rho)\big(d\theta\otimes d\theta + \sin^2(\theta)\,d\varphi\otimes d\varphi\big) \tag{8c}$$

is a 3-dimensional Riemannian metric on $\mathbb{R}^3$ with polar coordinates (ρ, θ, φ).

Problem 4

This problem continues the previous one.

Prove that ρ represents the geodesic distance with respect to h of the respective point with the origin $\rho = 0$.

Calculate all curvature components of h (best with respect to orthonormal frame using Cartan's structure equations; see GR-lecture last semester) and show that $(\mathbb{R}^3, h)$ is of constant sectional curvature -1. $(\mathbb{R}^3, h)$ is also a maximally symmetric manifold with 6-dimensional isometry group $\mathrm{Isom}(\mathbb{R}^3, h)$. What is that group? (You know it!) Characterise its subgroups $\mathrm{Isom}_p(\mathbb{R}^3, h)$ and $\mathrm{Isom}_{(p,u(p))}(\mathbb{R}^3, h)$ (compare Lecture 4).

This establishes (W, η, u) as an open (i.e. negatively curved space) FLRW-universe with flat spacetime! It is called the *Milne Universe*.

Problem 5

We are still in the Milne Universe with metric (8).

Show that a light signal sent by the comoving observer at position $\rho_1 = 0$ and eigentime τ_1 to the comoving observer at $\rho_0 > 0$ will be received there at eigentime

$$\tau_0 = \tau_1 \exp(\rho_0)\,. \tag{9}$$

Suppose the light signal is monochromatic with frequency ν_1 (measured with respect to proper time of observer at ρ_1). When received by the comoving observer at $\rho_0 > 0$ it will have frequency ν_0 (measured with respect to proper time of the observer at ρ_0). Calculate the redshift factor

$$z = \frac{\nu_1 - \nu_0}{\nu_0} \tag{10}$$

as a function of ρ_0 (the proper simultaneous distance).