Exercises for the lecture on

Special Topics in GR & Relativistic Cosmology

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Sheet 3

Problem 1

Consider the FLRW metric in coordinates (compare Lecture 4, equations (4.30-31))

$$g = c\mathrm{d}t \otimes c\mathrm{d}t - a^2(t)\Big(\mathrm{d}\chi \otimes \mathrm{d}\chi + \Sigma^2(\chi)\big(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\varphi \otimes \mathrm{d}\varphi\big)\Big)\,, \tag{1}$$

where

$$\Sigma(\chi) = \begin{cases} \sin(\chi) & \text{for } k = 1 \\ \chi & \text{for } k = 0 \\ \sinh(\chi) & \text{for } k = -1\,. \end{cases} \tag{2}$$

We are interested in the timelike geodesics.

Write down the Euler-Lagrange equation for the energy functional.

Prove that "radial" initial conditions with vanishing $\dot{\theta}$ and $\dot{\varphi}$ lead to constant values (equal to the initial ones) for (θ, φ); that is, the motion stays radial.

Argue why this solves the most general geodesic motion, even though the initial conditions seem special.

Prove that

$$pa = \text{const.}\,, \tag{3}$$

along the geodesic, where $p = p(\lambda)$ is the particle's 3-momentum with respect to the local observer defined by the cosmological four-velocity $u = \partial/\partial t$.

How do you interpret (3)? Does it violate momentum conservation in an expanding universe?

Problem 2

Consider the energy-momentum tensor of a perfect fluid by

$$T = (\rho + p/c^2)u \otimes u - g^{-1}p \tag{4}$$

or in components

$$T^{\alpha\beta} = (\rho + p/c^2)u^\alpha u^\beta - g^{\alpha\beta}p\,. \tag{5}$$

Here g^{-1} is the inverse metric, u the four-velocity of the fluid, ρ its rest-mass-density, and p its pressure in its rest frame. The metric is that of a FLRW spacetime:

$$g = \mathrm{d}x^0 \otimes \mathrm{d}x^0 - a^2(x^0)\hat{g}\,, \tag{6}$$

where $\hat{g}$ is of constant curvature $k \in \{1, -1, 0\}$. Calculate its covariant divergence $\nabla \cdot T$ (in components $\nabla_\alpha T^{\alpha\beta}$) and express it in a an orthonormal basis $\{e_0, e_1, e_2, e_3\}$ (dual basis $\{\theta^\alpha\}$) with respect to g in which $e_0 = u/c$. Recall that for that basis we explicitly calculated the connection 1-forms Lecture 4. For us here only the result $\omega^a{}_0 = (a_{,0}/a)\theta^a$ will be needed, where $a_{,0} = e_0(a) = \partial a/\partial x^0$. Alternatively, you can use the coordinate form using the Christoffel symbols that we calculated in Lecture 5. (You may convince yourself that the index-free calculation is shorter.)

Prove that the vanishing of the covariant divergence of (4) in the metric (6) is equivalent to

$$e_0(\rho a^3) + (p/c^2)e_0(a^3) = 0 \quad \text{and} \quad e_a(p) = 0\,. \tag{7}$$

Note that since $e_0 = \partial/\partial x^0$ this is equivalent to

$$(\rho a^3)^\cdot + (p/c^2)(a^3)^\cdot = 0 \quad \text{and} \quad p = p(t)\,. \tag{8}$$

It is interesting to note that the spatial constancy of ρ is *not* implied by these conditions.

Problem 3

Consider two metrics g and $\tilde{g}$ on space-time M which are conformally equivalent. This means that there is a positive function $a : M \to \mathbb{R}_+$ such that

$$g = a^2 \tilde{g}\,. \tag{9}$$

Let x^α be a local coordinate system with respect to which we express the components $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$ and their respective Christoffel symbols. Show that they are related by

$$\Gamma^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma} + \left(-\tilde{g}^{\alpha\lambda}\tilde{g}_{\beta\gamma}\frac{a_{,\lambda}}{a} + \delta^\alpha_\beta \frac{a_{,\gamma}}{a} + \delta^\alpha_\gamma \frac{a_{,\beta}}{a}\right) \tag{10}$$

Here we write as usual $a_{,\alpha} := \partial a/\partial x^\alpha$, etc.

Let $z : \mathbb{R} \supseteq I \to M$ be a lightlike geodesic on (M, g). Show that the geodesic equation for its coordinate representation $z^\alpha = x^\alpha \circ z$, $\lambda \mapsto z^\alpha(\lambda)$, is equivalent to

$$\ddot{z}^\alpha + \left(\tilde{\Gamma}^\alpha_{\beta\gamma} \circ z\right)\dot{z}^\beta\,\dot{z}^\beta = -2\frac{(a \circ z)^\cdot}{(a \circ z)}\dot{z}^\alpha\,. \tag{11}$$

Now consider a reparametrisation diffeomorphism

$$f : \mathbb{R} \supseteq I \to I' \subseteq \mathbb{R} \tag{12}$$

and the corresponding reparametrised curve

$$y := z \circ f^{-1}\,. \tag{13}$$

Show that (11) is equivalent to

$$\ddot{y}^\alpha + \left(\tilde{\Gamma}^\alpha_{\beta\gamma} \circ y\right)\dot{y}^\beta\,\dot{y}^\beta = \dot{y}^\alpha \left[\frac{\ddot{h}}{\dot{h}} - 2\frac{(a \circ y)^\cdot}{(a \circ y)}\right], \tag{14}$$

where $h := f^{-1}$.

Problem 4

Use the results of Problem 2 to prove the following theorem: The curve z is a lightlike geodesic for (M, g) if and only if the reparametrised curve $y = z \circ f^{-1}$ is a lightlike geodesic for $(M, \tilde{g})$ where the reparametrisation map $f : I \ni \lambda \mapsto \tilde{\lambda} := f(\lambda) \in \tilde{I}$ and its inverse $h := f^{-1} : \tilde{I} \ni \tilde{\lambda} \mapsto \lambda := h(\tilde{\lambda}) \in I$ satisfie any of the two equivalent conditions (show the equivlence):

$$\frac{\ddot{f}}{\dot{f}} = -2\frac{(a \circ z)^{\cdot}}{(a \circ z)} \qquad \Longleftrightarrow \qquad \frac{\ddot{h}}{\dot{h}} = 2\frac{(a \circ y)^{\cdot}}{(a \circ y)}\,. \tag{15}$$

which in turn are are equivalent to

$$\tilde{\lambda} = f(\lambda) = C\int_{\lambda_0}^{\lambda} \frac{d\lambda'}{a^2(z(\lambda'))} \qquad \Longleftrightarrow \qquad \lambda = h(\tilde{\lambda}) = \tilde{C}\int_{\tilde{\lambda}_0}^{\tilde{\lambda}} d\tilde{\lambda}'\, a^2(y(\tilde{\lambda}'))\,. \tag{16}$$

Problem 5

We now apply the foregoing "warped-product" metrics of FLRW form

$$g = dx^0 \otimes dx^0 - a^2(x^0)\hat{g}_{ab}\, dx^a \otimes dx^b\,, \tag{17}$$

where $\hat{g}$ is independent of x^0 and not necessarily of constant curvature (hence this class is more general than FLRW).

We write this in the form (9) with

$$\tilde{g} = d\eta \otimes d\eta - \hat{g}\,, \tag{18}$$

where the coordinate η is defined in terms of the coordinate x^0 by

$$\eta(x^0) = \int_c^{x^0} \frac{dx'^0}{a(x'^0)} \tag{19}$$

and is called the *conformal time*.

Show that the geodesics of $\tilde{g}$ are given by $\tilde{\lambda} \mapsto (\eta(\tilde{\lambda}), \vec{x}(\tilde{\lambda}))$, where $\eta(\tilde{\lambda}) = a\tilde{\lambda} + b$ and $\vec{x}(\tilde{\lambda})$ is a geodesic of $\hat{g}$. Hence for $a \neq 0$ we conclude that along such a geodesic η grows proportional to the affine parameter, i.e. is itself an affine parameter. Moreover, if the geodesic is lightlike this parameter is affinely equivalent to the proper length in $(\hat{M}, \hat{g})$ (compare Lecture 5).

Use this and the second equation in (16) to derive the following expression for the value of the affine parameter λ along a lightlike geodesic $\lambda \mapsto (x^0(\lambda), \vec{x}(\lambda))$ at the instant it hits the constant-time hypersurface x^0:

$$\lambda(x^0) = C\int_k^{x^0} dx'^0 a(x'^0)\,. \tag{20}$$