Exercises for the lecture on

Special Topics in GR & Relativistic Cosmology

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Sheet 4

Problem 1

In Lecture 7 we discussed that for $\Omega_{\text{rad}} = \Omega_\Lambda = 0$ the relation between luminosity-distance d_L and redshift z can be expressed in closed form via Mattig's equation

$$d_L(z) = \frac{2c}{H_0} \cdot \frac{(\Omega - 2)\left[\sqrt{1+z\Omega} - 1\right] + z\Omega}{\Omega^2}\,. \tag{1}$$

We proved this relation for the "closed" case $k = 1$. Show that this formula also holds in the other cases $k = 0$ (flat) and $k = -1$ ("open").

Problem 2

Prove that equation (1) is equivalent to

$$d_L(z) = \frac{c}{H_0} \cdot z \cdot \frac{1 + \sqrt{1+z\Omega} + z}{1 + \sqrt{1+z\Omega} + z\Omega/2} \tag{2}$$

in which numerator and denominator stay finite for $\Omega \to 0$.

Problem 3

An object of linear extension D perpendicular to the line of sight is seen from a distance much larger than D under an angle

$$\Delta\varphi(D, z) = \frac{D}{d_A(z)} \tag{3}$$

where $d_A(z) = d_L(z)/(1+z)^2$ is the angular distance. Specialise to the case $\Omega_{\text{rad}} = \Omega_\Lambda = 0$, in which Mattig's formula applies, and show that the function $z \mapsto \varphi(D, z)$ (which we only consider for positive z) is $\propto z^{-1}$ for small Z and $\propto z^{+1}$ for large z. Hence the the function must have a minimum at some finite z-value z_*. Determine z_* for the flat case $k = 0$.

Relate this to the "Scheinriese" *Herrn Tur Tur* whom *Jim Knopf und Lukas der Lokomotivfüherer* encounter in the desert *Ende der Welt* (if you happen to know that story).

Problem 4

Typical current values for the cosmological Ω-parameters are given by

$$\begin{aligned} \Omega_{\text{dust}} &= 0.27\,, \\ \Omega_{\text{rad}} &= 8.25 \times 10^{-5}\,, \\ \Omega_{\Lambda} &= 0.73\,, \\ \Omega_{\text{k}} &= \text{compatible with } 0\,. \end{aligned} \tag{4}$$

Calculate the redshift at which the "radiation" (relativistic matter) contribution equals that of "dust". What would then be their ratio to the Λ-contribution? How does that redshift compare to that of "recombination", which is approximately $z_{\text{rc}} = 1100$? What lessons can we draw from that as regards neglecting Ω_{rad}?

Problem 5

In Lecture 6 we wrote the showed how to write the Friedmann equation in the dimensionless form

$$(dx/d\lambda)^2 + V(x) = E\,, \tag{5}$$

where

$$x = a/H_0\,, \quad \lambda = tH_0\,, \quad E = \Omega_k\,, \tag{6}$$

and

$$V(x) = -\frac{\Omega_{\text{rad}}}{x^2} - \frac{\Omega_{\text{dust}}}{x} - \Omega_\Lambda\, x^2\,. \tag{7}$$

Find all solutions to (5) for $\Omega_{\text{rad}} = \Omega_k = 0$ in closed analytic form. Choose the constants of integration in such a way that $x(\lambda = 0) = 0$.

Hint: You have to distinguish the cases $\Omega_\Lambda > 0$, $\Omega_\Lambda = 0$, and $\Omega_\Lambda < 0$.

For positive Λ derive a formula for the *age of the universe*, that is the time $t = t_0$ at which $x(t_0) = 1$ (corresponding to $a(t_0) = a_0$). Express that age as multiple of the Hubble-time $1/H_0$, where $H_0 =: h \cdot 100 \cdot km \cdot s^{-1} \cdot Mpc$. Calculate the age of the universe in years for the Ω values (4) and $h = 0.72$. Finally, determine the x-value and the redshift at which $\ddot{x}$ changes sign. What does that value tell you?