Exercises for the lecture on

Special Topics in GR & Relativistic Cosmology

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Sheet 6

Problem 1

We recall the“middle” Friedmann equation from Lecture 6 (formula (6.16b) in the manuscript)

$$\dot{a}^2(t) = \frac{8\pi G}{3}\rho(t)a^2(t) + \frac{\Lambda c^2}{3}a^2(t) - kc^2\,. \tag{1}$$

Like we did for time $t = t_0$, we introduce the following Ω-parameters, now for at any time t so that they become functions of t:

$$\Omega_m(t) := \frac{8\pi G\rho(t)}{3H^2(t)}\,, \tag{2a}$$

$$\Omega_\Lambda(t) := \frac{\Lambda c^2}{3H^2(t)}\,, \tag{2b}$$

$$\Omega_k(t) := \frac{-kc^2}{H^2(t)a^2(t)}\,. \tag{2c}$$

Equation (1) than merely says that the sum of these equals 1. We also have $\rho = \rho_{\text{rad}} + \rho_{\text{dust}}$, with $\rho_{\text{rad}}a^4 = const.$ and $\rho_{\text{dust}}a^3 = const.$.

Derive the following formulae relating the Ω parameter at time t, when the scale-factor is $a(t)$, to those that time t_0, when the scale-factor is $a(t_0)$. In what follows we write Ω for $\Omega(t)$, Ω^0 for $\Omega(t_0)$, a for $a(t)$, and a_0 for $a(t_0)$.

$$\Omega_{\text{rad}} = \frac{\Omega^0_{\text{rad}}}{\Omega^0_{\text{rad}} + \Omega^0_{\text{dust}}\cdot(a/a_0) + \Omega^0_k\cdot(a/a_0)^2 + \Omega^0_\Lambda\cdot(a/a_0)^4}\,, \tag{3a}$$

$$\Omega_{\text{dust}} = \frac{\Omega^0_{\text{dust}}}{\Omega^0_{\text{rad}}\cdot(a_0/a) + \Omega^0_{\text{dust}} + \Omega^0_k\cdot(a/a_0) + \Omega^0_\Lambda\cdot(a/a_0)^3}\,, \tag{3b}$$

$$\Omega_k = \frac{\Omega^0_k}{\Omega^0_{\text{rad}}\cdot(a_0/a)^2 + \Omega^0_{\text{dust}}\cdot(a_0/a) + \Omega^0_k + \Omega^0_\Lambda\cdot(a/a_0)^2}\,, \tag{3c}$$

$$\Omega_\Lambda = \frac{\Omega^0_\Lambda}{\Omega^0_{\text{rad}}\cdot(a_0/a)^4 + \Omega^0_{\text{dust}}\cdot(a_0/a)^3 + \Omega^0_k\cdot(a_0/a)^2 + \Omega^0_\Lambda}\,. \tag{3d}$$

Problem 2

In Lecture 10 we mentioned the Plack-Mass and Planck-Langth. Together with the Planck-Time and Planck-Temperature, they form what is known as the “Planck-Units”

$$m_p := \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8}\,\mathrm{kg} = 1.221 \times 10^{19}\,\mathrm{GeV}/c^2\,, \tag{4a}$$

$$\ell_p := \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35}\,\mathrm{m}\,, \tag{4b}$$

$$t_p := \sqrt{\frac{\hbar G}{c^5}} = 5.391 \times 10^{-44}\,\mathrm{s}\,, \tag{4c}$$

$$T_p := \sqrt{\frac{\hbar c^5}{G k_B^2}} = 1.4168 \times 10^{32}\,\mathrm{K}\,. \tag{4d}$$

Note that with a Hubble-Radius

$$R_H := c/H_0 = 13.7 \times 10^9\,\mathrm{ly} = 1.3 \times 10^{26}\,\mathrm{m} \tag{5}$$

the geometric mean between the Hubble radius and the Planck-Length is 4.5×10^{-5} m, which is close to the average size of a human cell. Hence there are as many orders of magnitude between the size of the visible universe (Hubble-Radius) and the human cell as there are between the size of the cell and the Planck-Length.

Suppose the universe evolved from the Big-Bang at $t = 0$ to t_P according to our equations – clearly an outrageous hypothesis; but we make it anyway. Using equations (3), show that one may safely set $\Omega_{\mathrm{rad}} = 1$ and forget about all other Ωs in oder to describe the evolution during that period.

Use the Friedmann equation (1) to deduce that

$$\left(\frac{a_p}{a_0}\right)^2 = 2 \cdot \sqrt{\Omega^0_{\mathrm{rad}}} \cdot H_0 \cdot t_p = 2 \cdot \sqrt{\Omega^0_{\mathrm{rad}}} \cdot \frac{\ell_p}{R_H} \tag{6}$$

Use this and (3c) to further derive:

$$\frac{\Omega_k(t_P)}{\Omega_k(t_0)} = \frac{2}{\sqrt{\Omega^0_{\mathrm{rad}}}} \cdot \frac{\ell_p}{R_H}\,. \tag{7}$$

On Problem-Sheet 4, Problem 4, we saw that $\Omega^0_{\mathrm{rad}} = 8.25 \times 10^{-5} \approx 10^{-4}$. Hence (7) tells us that if $\Omega_k(t)$ is close to zero at $t = t_0$ (now) to some accuracy $\varepsilon \ll 1$, it must have been close to zero at the Planck-Time with an accuracy of 10^{-59}. This is, in more quantitative detail, the "flatness-problem" that we already mentioned in Lecture 10. Discuss: Why, precisely, is that considered to be a problem?

Problem 3

Suppose an exponential expansion

$$a(t) = a_0 \cdot \exp\big(H(t - t_0)\big)\,. \tag{8}$$

How does Ω_k, defined in (2c), depend on t under that evolution? If (8) holds for $t < t_p$, how many e-folds are needed in order to suppress a generic initial Ω_k of order one by the needed factor of roughly 10^{-60} derived in the previous problem? (An "e-fold" is the time needed to increase an exponentially growing quantity by a factor of e.)