Sheet 1 : Solutions

Problem 1

Poisson's Equation

$$\Delta \phi = 4\pi G \varrho \tag{1.1.1}$$

has a unique solution obeying

$$\phi(\|\vec{x}\| \to \infty) = 0 \tag{1.1.2}$$

given by

$$\phi(t, \vec{x}) = -G \int \frac{\varrho(t, \vec{x}')}{\|\vec{x} - \vec{x}'\|} \, d^3x' \tag{1.1.3}$$

provided $\varrho(t, \cdot)$ falls-off faster as r^{-2} for $r := \|\vec{x}\| \to \infty$, as it must if the integral of ϱ, i.e. the total mass, is to be finite.

Note the meaning of "falling off faster": A real valued function $f: \mathbb{R}^3 \to \mathbb{R}$ is said to fall of faster than r^{-n}, $n \in \mathbb{N}$, at infinity, i.e. for $\|\vec{x}\| \to \infty$, iff there exists a compact set $K \subset \mathbb{R}^3$ and positive numbers $c, \varepsilon \; \mathbb{R}_{>0}$, so that

$$|f(\vec{x})| < \frac{c}{r^{n+\varepsilon}} \qquad (1.1.4)$$

for all $\vec{x} \in \mathbb{R}^3 - K$, i.e. outside the compact set K. In other words: $|f(\vec{x})|$ is bounded above by $c / r^{n+\varepsilon}$ in a neighbourhood of infinity.

On the other hand, if no conditions at infinity are posed on $\mathbb{R}^3$ than any solution to Poissons equation can be modified by adding elements of the kernel of Δ; but $\mathrm{Ker}(\Delta)$ is then huge; e.g., let

$$\psi(\vec{x}) = t_{a_1 \dots a_n} x^{a_1} \dots x^{a_n} \qquad (1.1.5)$$

where t is a symmetric trace-free tensor on $\mathbb{R}^3$ that is totally trace free

$$t_{a_1 \dots a_n} = t_{(a_1 \dots a_n)} \qquad (1.1.6)$$

$$\delta^{a_1 a_2} t_{a_1 a_2 a_3 \dots a_n} = 0 \qquad (1.1.7)$$

Then

$$\Delta \psi(\vec{x}) = n(n-1) t^{a}{}_{b a_3 \dots a_n} = 0 \qquad (1.1.8)$$

If $\rho = k =$ const, then solutions of

$$\Delta \phi = 4\pi G \rho$$

are

$$\phi = \frac{2}{3}\pi G \, \|\vec{x}\|^2 + \vec{V}\cdot\vec{x} + \phi_0 + \psi \qquad (1.1.9)$$

where $\vec{V} \in \mathbb{R}^3$ is a constant vector, $\phi_0 \in \mathbb{R}$ also constant, and ψ any element in $\mathrm{Ker}(\Delta)$.

Note: Even though the source ρ is homogeneous, i.e.

$$\left.\begin{aligned} &\rho(\vec{x}+\vec{c}) = \rho(\vec{x}) \quad \forall \vec{c} \in \mathbb{R}^3 \\ &\Leftrightarrow \quad \vec{\nabla}\rho = 0 , \end{aligned}\right\} \quad (1.1.10)$$

ϕ is <u>not</u> homogeneous

$$\vec{\nabla}\phi = \frac{4\pi}{3} G \vec{x} + \vec{V} + \vec{\nabla}\psi \qquad (1.1.11)$$

where $\partial_a \psi$ is a sum of terms of the form $\pm\, a_{a a_2 \dots a_n} x^{a_2} \dots x^{a_n}$.

The situation is totally different if the domain of ϕ is closed, i.e. compact without boundary, like S^3. Then, by Gauß' theorem

$$\int_{S^3} \phi \, \Delta \phi \, d\mu = \int_{S^3} \phi \, \vec{\nabla} \cdot \vec{\nabla} \phi$$

$$= \underbrace{\int_{S^3} \vec{\nabla} (\phi \vec{\nabla} \phi) \, d\mu}_{\int_{\partial S^3} (\phi \vec{\nabla} \phi) \cdot \vec{n} \, do \, = \, 0} - \int_{S^3} \| \vec{\nabla} \phi \|^2 \, d\mu$$

since $\partial S^3 = \emptyset$.

Hence

$$\int_{S^3} \phi \, \Delta \phi \, d\mu = - \int_{S^3} \| \vec{\nabla} \phi \|^2 \, d\mu$$

from which we immediately infer:

$$\Delta \phi = 0 \iff \phi = \text{const.}$$

Moreover, also by Gauß' law

$$\int_{S^3} \Delta \phi \, d\mu = \int_{\partial S^3} (\vec{\nabla} \phi) \cdot \vec{n} \, d\mu = 0$$

since $\partial S^3 = \emptyset$.

Hence solutions to

$$\Delta \phi = 4\pi G \rho$$

cannot exist if

$$\int_{S^3} \rho \, d\mu \neq 0$$

If that integral is zero, solutions exist and are unique (like for a distribution of electric charges the total sum of which is zero). For mass distributions $\rho \geq 0$ no solution exists except $\phi = \text{const}$ for $\rho \equiv 0$.

Problem 2

$$m\ddot{\vec{X}}_a(t) = G\sum_{\substack{b=1\\b\neq a}}^{N} m_a m_b \frac{\vec{X}_b(t)-\vec{X}_a(t)}{\|\vec{X}_b(t)-\vec{X}_a(t)\|^3} \quad (1.2.1)$$

If

$$V(\vec{X}_1 \cdots \vec{X}_N) = -\frac{1}{2}\sum_{\substack{a,b=1\\a\neq b}}^{N} \frac{G m_a m_b}{\|\vec{X}_a-\vec{X}_b\|} \quad (1.2.2)$$

then

$$\vec{\nabla}_c V = -\frac{G}{2}\left\{\sum_{\substack{b=1\\b\neq c}}^{N} \vec{\nabla}_c \frac{m_c m_b}{\|\vec{X}_c-\vec{X}_b\|} + \sum_{\substack{a=1\\a\neq c}}^{N} \vec{\nabla}_c \frac{m_a m_c}{\|\vec{X}_a-\vec{X}_c\|}\right\} \quad (1.2.3)$$

But both sums are the same;
hence

$$\vec{\nabla}_a V = -\vec{\nabla}_a \sum_{\substack{b=1\\b\neq a}}^{N} G \frac{m_a m_b}{\|\vec{X}_a-\vec{X}_b\|}$$

$$= \sum_{\substack{b=1\\b\neq a}}^{N} G m_a m_b \frac{\vec{X}_a-\vec{X}_b}{\|\vec{X}_a-\vec{X}_b\|^3} \quad . \quad (1.2.4)$$

$V(\vec{x}_1 \cdots \vec{x}_N)$ is obviously a homogeneous function of degree -1:

$$V(\lambda \vec{x}_1, \cdots, \lambda \vec{x}_N) = \lambda^{-1} V(\vec{x}_1, \cdots, \vec{x}_N) \quad (1.2.5)$$

Hence

$$\frac{d}{d\lambda}\Big|_{\lambda=1} V(\lambda \vec{x}_1, \cdots, \lambda \vec{x}_N)$$

$$= \sum_{a=1}^{N} \vec{x}_a \cdot \vec{\nabla}_a V(\vec{x}_1, \cdots, \vec{x}_N)$$

$$= -\frac{1}{\lambda^2}\Big|_{\lambda=1} V(\vec{x}_1, \cdots, \vec{x}_N)$$

$$= -V(\vec{x}_1, \cdots, \vec{x}_N) \quad (1.2.6)$$

Note: for

$$V(\lambda \vec{x}_1, \cdots, \lambda \vec{x}_N) = \lambda^{n} V(\vec{x}_1, \cdots, \vec{x}_N) \quad (1.2.7)$$

the same reasoning proves Euler's theorem:

$$\sum_{a=1}^{N} \vec{x}_a \cdot \vec{\nabla}_a V = n V. \quad (1.2.8)$$

Problem 3

We seek solutions of (1.2.1) which are of the form

$$\vec{x}_a(t) = a(t)\,\vec{y}_a \qquad (1.3.1)$$

where $(\vec{y}_1, \dots, \vec{y}_N)$ are N fixed (time independent) vector and $a : \mathbb{R} \supseteq I \to \mathbb{R}$.

Introducing (1.3.1) into (1.2.1) gives

$$m_a \ddot{a}\,\vec{y}_a = \frac{1}{a^2} \sum_{\substack{b=1 \\ b\neq a}}^{N} G m_a m_b \frac{\vec{y}_b - \vec{y}_a}{\|\vec{y}_b - \vec{y}_a\|^3} \qquad (1.3.2)$$

or

$$\ddot{a}\, a^2 m_a \vec{y}_a = G \sum_{\substack{b=1 \\ b\neq a}}^{N} m_a m_b \frac{\vec{y}_b - \vec{y}_a}{\|\vec{y}_b - \vec{y}_a\|^3} \qquad (1.3.3)$$

Since the right-hand side is independent of t, so must be the left-hand side; hence

$$\ddot{a}\, a^2 = C = \text{const.} \qquad (1.3.4)$$

$$\ddot{a} = \frac{c}{a^2} \quad \Big| \cdot \dot{a}$$

$$\frac{1}{2}\left(\dot{a}^2\right)^{\cdot} = \left(-\frac{c}{a}\right)^{\cdot}$$

$$\Rightarrow \quad \frac{1}{2}\dot{a}^2 + \frac{c}{a} = E = \text{const.} \qquad (1.3.5)$$

Note that the right-hand side of (1.3.3) equals

$$-\vec{\nabla}_a V \Big|_{\vec{x}_a = \vec{y}_a} \qquad (1.3.6)$$

Hence if we multiply (1.3.3) by $\vec{y}_a$ (scalar-multiplication) and sum over a, we get with (1.3.4)

$$c \cdot \sum_{a=1}^{N} m_a \|\vec{y}_a\|^2 = -\sum_{a=1}^{N} \vec{y}_a \cdot \vec{\nabla}_a V \Big|_{\vec{x}=\vec{y}}$$

$$= V(\vec{y}_1, \dots, \vec{y}_N)$$

$$= -\frac{G}{2} \sum_{\substack{a,b=1 \\ a \neq b}}^{N} \frac{m_a m_b}{\|\vec{y}_a - \vec{y}_b\|} < 0 \qquad (1.3.7)$$

$$\curvearrowright \quad c = \frac{V(\vec{y}_1, \dots, \vec{y}_N)}{\sum_{a=1}^{N} m_a \|\vec{y}_a\|^2} < 0 \qquad (1.3.8)$$

Problem 4

We wish to determine all solutions to

$$\frac{1}{2}\dot{a}^2 + \frac{C}{a} = E \qquad (1.4.1)$$

for $C < 0$ and E either < 0, $= 0$, or > 0.

We write $C = -k$, $k > 0$, and restrict to positive a (without loss of generality). Then

$$\dot{a}^2 = 2\left(\frac{k}{a} + E\right) \qquad (1.4.2)$$

1. Case: $E = 0$

$$\dot{a} = \pm\sqrt{2k}\; a^{-1/2}$$

$$a^{1/2}\, da = \pm\sqrt{2k}\; dt$$

$$\frac{2}{3}\left(a^{3/2} - a_0^{3/2}\right) = \pm\sqrt{2k}\,(t - t_0)$$

Take t non-negative and $a = 0$ for $t = 0$, then

$$a(t) = 3\cdot\sqrt{\frac{k}{2}}\quad t^{2/3} \qquad (1.4.3)$$

2. case : $E = -\varepsilon < 0$

Then (1.4.2) reads

$$\dot{a}^2 = 2\left(\frac{k}{a} - \varepsilon\right) \tag{1.4.4}$$

so that there is a maximal a:

$$\frac{k}{a} - \varepsilon \geq 0 \iff a \leq \frac{k}{\varepsilon} \tag{1.4.5}$$

If instead of t we use the parameter

$$\lambda = \sqrt{\varepsilon}\, t \tag{1.4.6}$$

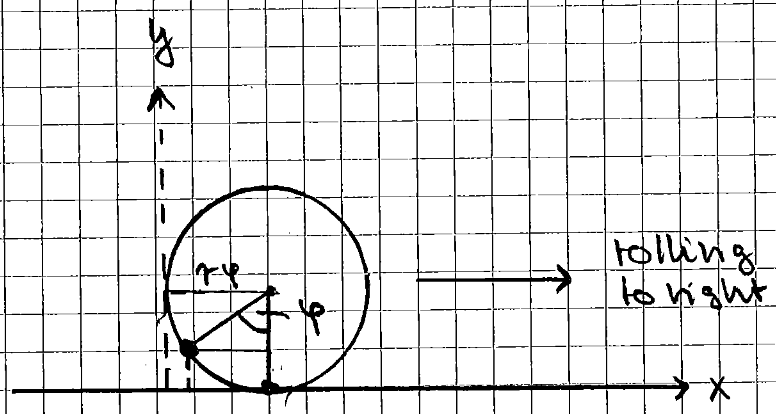
then (1.4.4) reads

$$\left(\frac{da}{d\lambda}\right)^2 = \frac{2r}{a} - 1 \tag{1.4.7a}$$

with

$$r := \frac{k}{\varepsilon} \tag{1.4.7b}$$

But (1.4.7) is just the differential equation for a <u>cycloid</u>



Radius of wheel : r

Rolling angle : φ

The x and y coordinates of point on the circumference, that is at origin for $\varphi = 0$ is

$$x(\varphi) = r(\varphi - \sin\varphi) \qquad (1.4.8a)$$

$$y(\varphi) = r(1 - \cos\varphi) \qquad (1.4.8b)$$

Hence

$$\frac{dy}{dx} = \frac{\sin\varphi}{1-\cos\varphi}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1-\cos^2\varphi}{(1-\cos\varphi)^2}$$

$$= \frac{1+\cos\varphi}{1-\cos\varphi} = \frac{2}{1-\cos\varphi} - 1$$

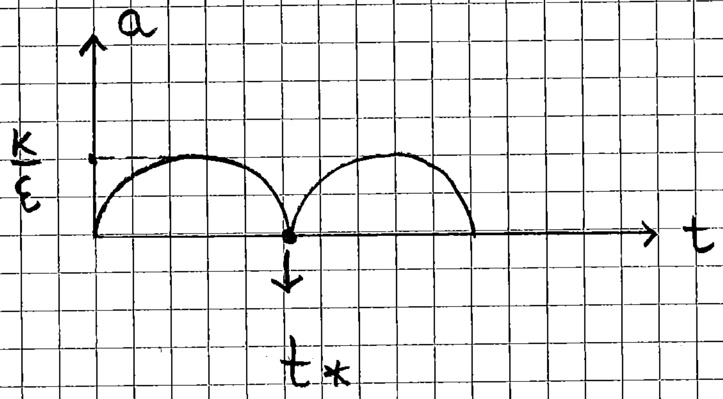
$$= \frac{2r}{y} - 1 \qquad (1.4.9)$$

E1.13

Comparing (1.4.9) with (1.4.7) then allows to apply (1.4.8) as solution with $y = a$, $x = \lambda$

$$\lambda(\varphi) = \frac{K}{\varepsilon}(\varphi - \sin\varphi) \qquad (1.4.10a)$$

$$a(\varphi) = \frac{K}{\varepsilon}(1 - \cos\varphi) \qquad (1.4.10b)$$



$$t_* = \frac{1}{\sqrt{\varepsilon}}\,\lambda(2\pi) = \frac{2\pi K}{\varepsilon^{3/2}} \qquad (1.4.11)$$

= Time from Big Bang ($\varphi = 0$) to Big Crunch ($\varphi = 2\pi$).

3. Case: $E = \varepsilon > 0$

Now (1.4.2) reads

$$\dot{a}^2 = 2\left(\frac{k}{a} + \varepsilon\right) \qquad (1.4.12)$$

Again we use (1.4.6) for λ and rewrite this as

$$\left(\frac{da}{d\lambda}\right)^2 = \frac{2r}{a} + 1 \qquad (1.4.13)$$

with $r = \frac{k}{\varepsilon}$

Again this corresponds to the equation similar to that for a cycloid, though I do not know whether this curve has a name:

Consider

$$x(\varphi) = r(\sinh\varphi - \varphi) \qquad (1.4.15a)$$

$$y(\varphi) = r(\cosh\varphi - 1) \qquad (1.4.15b)$$

Then

$$\frac{dy}{dx} = \frac{\sinh\varphi}{\cosh - 1}$$

E1.15

$$\left(\frac{dy}{dx}\right)^2 = \frac{\cosh^2\varphi - 1}{(\cosh\varphi - 1)^2}$$

$$= \frac{\cosh\varphi + 1}{\cosh\varphi - 1}$$

$$= \frac{2 + (\cosh\varphi - 1)}{\cosh\varphi - 1}$$

$$= \frac{2r}{y} + 1 \qquad (1.4.16)$$

Hence as solution we get

$$\lambda(\varphi) = \frac{k}{\varepsilon}(\sinh\varphi - \varphi) \qquad (1.4.17a)$$

$$a(\varphi) = \frac{k}{\varepsilon}(\cosh\varphi - 1) \qquad (1.4.17b)$$

where $\varphi \in [0, \infty)$. For $t = 0$, corresponding to $\varphi = 0$ and $a = 0$ (Big Bang), the system expands forever and for $\varphi \to \infty$ asymptotically (where $\lambda(\varphi) = (k/\varepsilon)\sinh\varphi$, etc.)

$$a = \frac{k}{\varepsilon}\cosh\varphi - \frac{k}{\varepsilon}\left(1 + \sinh^2(\varphi)\right)^{1/2}$$

$$= \frac{k}{\varepsilon}\left(1 + \lambda^2\frac{\varepsilon^2}{k^2}\right)^{1/2}$$

$$= \frac{k}{\varepsilon}\left(1 + \frac{\varepsilon^3}{k^2}t^2\right)^{1/2} \cong \sqrt{\varepsilon}\, t \qquad (1.4.18)$$

Problem 5

Central configurations are defined as solutions of

$$c\, m_a \vec{y}_a = G \sum_{b \neq a} m_a m_b \frac{\vec{y}_b - \vec{y}_a}{\|\vec{y}_b - \vec{y}_a\|^3} \qquad (1.5.1)$$

for some – necessarily negative – c, that according to (1.3.8) turns out to be

$$c = \frac{V(\vec{y}_1, \dots, \vec{y}_N)}{\sum_a m_a \|\vec{y}_a\|^2} \qquad (1.5.2)$$

If $(\vec{y}_1, \dots, \vec{y}_N)$ is a solution, then so is $(\vec{y}\,'_1, \dots, \vec{y}\,'_N)$ with

$$\vec{y}\,'_a = \lambda \vec{y}_a \qquad \lambda > 0 \qquad (1.5.3)$$

for

$$c' = \lambda^{-3} c \,. \qquad (1.5.4)$$

Hence, w.l.o.g., we may assume $\sum m_a \|\vec{y}_a\|^2 = 1$, in which case $c = V(\vec{y}_1, \dots, \vec{y}_N)$.
It is obvious from (1.5.1) that then $(\vec{y}_1, \dots, \vec{y}_N)$ solution $\Rightarrow (D\vec{y}_1, \dots, D\vec{y}_N)$ solution for all $D \in SO(3)$; note
$\|D\vec{y}_b - D\vec{y}_a\| = \|D(\vec{y}_a - \vec{y}_b)\| = \|\vec{y}_b - \vec{y}_a\|$.

Problem 6

Since $C < 0$ we call $k = -C$ and write the condition for central configuration as

$$k\, m_a \vec{y}_a + \sum_{\substack{b=1 \\ b \neq a}}^{N} G\, m_a m_b \frac{\vec{y}_b - \vec{y}_a}{\|\vec{y}_b - \vec{y}_a\|^3} = 0 \quad (1.6.1)$$

Summation over a gives with

$$\sum_{a=1}^{N} \sum_{\substack{b=1 \\ b \neq a}}^{N} m_a m_b \frac{\vec{y}_b - \vec{y}_a}{\|\vec{y}_b - \vec{y}_a\|^3}$$

$$= \frac{1}{2} \sum_{\substack{a,b=1 \\ a \neq b}}^{N} \underbrace{m_a m_b}_{\text{symmetric}} \underbrace{\frac{\vec{y}_b - \vec{y}_a}{\|\vec{y}_b - \vec{y}_a\|^3}}_{\text{antisymmetric}} \quad (1.6.2)$$

under $(a \leftrightarrow b)$

$$= 0$$

$$\rightarrow \sum_a m_a \vec{y}_a = 0 \quad (1.6.3)$$

E1.18

Let

$$M := \sum_{a=1}^{N} m_a \qquad (1.6.4)$$

then we wish to show the identity

$$m_a \vec{y}_a = \frac{1}{M} \sum_{\substack{b=1 \\ b \neq a}}^{N} m_a m_b (\vec{y}_a - \vec{y}_b)$$

$$+ \frac{m_a}{M} \sum_{b=1}^{N} m_b \vec{y}_b \qquad (1.6.5)$$

But the right-hand side is

$$\frac{1}{M} m_a \vec{y}_a \sum_{b \neq a} m_b - \frac{m_a}{M} \sum_{b \neq a} m_b \vec{y}_b$$

$$+ \frac{m_a}{M} \sum_{b} m_b \vec{y}_b$$

$$= \frac{1}{M} m_a \vec{y}_a (M - m_a) + \frac{m_a}{M} m_a \vec{y}_a$$

$$= m_a \vec{y}_a \qquad (1.6.6)$$

For configurations satisfying (1.6.3) we may thus set

$$m_a \vec{y}_a = \frac{1}{M} \sum_{\substack{b=1 \\ b \neq a}}^{N} m_a m_b (\vec{y}_a - \vec{y}_b) \qquad (1.6.7)$$

on the left of (1.6.1), which then

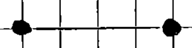
turns into

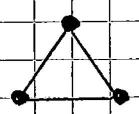
$$\sum_{\substack{b=1\\ b\neq a}}^{N} m_a m_b \left(\frac{K}{M} - \frac{G}{r_{ab}^{3}}\right)(\vec{y}_a - \vec{y}_b) = 0 \qquad (1.6.8)$$

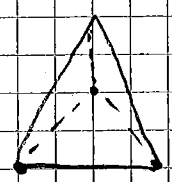
which is fullfilled for

$$r_{ab} = \left(\frac{GM}{K}\right)^{1/3} \qquad (1.6.9)$$

i.e. equal mutual distances irrespectively of mass-ratios.

2)  2 masses

3)  equilateral triangle

4)  tetrahedron

Note the following analogy. Let $(e_1, \dots, e_N)$ be N positive electric charges which are confined to stay on a sphere of radius R in $\mathbb{R}^3$. What is their configuration of minimal potential energy?

E1.20

The total potential energy is

$$V = \frac{1}{8\pi\varepsilon_0} \sum_{\substack{a,b=1 \\ a\neq b}}^{N} \frac{e_a e_b}{\|\vec{y}_a - \vec{y}_b\|} \qquad (1.6.10)$$

The constraints (Nebenbedingungen) are

$$\phi_a(\vec{y}_a) = \|\vec{y}_a\|^2 - R^2 = 0 \qquad (1.6.11)$$

Use Lagrange multiplier λ_a to implement them and extremize

$$\psi(\vec{y}_1, \dots, \vec{y}_N) = V + \sum_a \lambda_a \phi_a \qquad (1.6.12)$$

$$\nabla_{\vec{y}_a} \psi = 0 \iff$$

$$\underbrace{-\frac{1}{4\pi\varepsilon_0} \sum_{\substack{b=1 \\ b\neq a}}^{N} e_a e_b \frac{\vec{y}_a - \vec{y}_b}{\|\vec{y}_a - \vec{y}_b\|^3}}_{\vec{\nabla}_a V} + 2\lambda_a \vec{y}_a = \vec{0} \qquad (1.6.13)$$

$$\rightsquigarrow \underbrace{\sum \vec{y}_a \vec{\nabla}_a V}_{-V} + 2\lambda_a \underbrace{\|\vec{y}_a\|^2}_{R^2} = 0$$

$$\rightsquigarrow \lambda_a = V/2R^2 =: C/2$$

$$\rightarrow C\vec{y}_a = \frac{1}{4\pi\varepsilon_0} \sum_{b\neq a} e_a e_b \frac{\vec{y}_b - \vec{y}_a}{\|\vec{y}_b - \vec{y}_a\|^3} . \qquad (1.6.14)$$

This is close but not quite identical to

$$c\, m_a \vec{y}_a = G \sum_{b \neq a} m_a m_b \frac{\vec{y}_b - \vec{y}_a}{\|\vec{y}_b - \vec{y}_a\|^3}. \qquad (1.6.15)$$

Not quite because there is no e_a on the left hand side of (1.6.14). For equal masses though the central configuration problem is identical to that of minimizing the potential energy of equal charges on a 2-sphere.