Sheet 2: Solutions

Problem 1

$$\dot{\vec{x}} = \frac{\dot{R}}{R}\vec{x} \quad \text{(Hubble flow)} \tag{2.1.1}$$

$$\rightsquigarrow \ddot{\vec{x}} = \left(\frac{\dot{R}}{R}\right)^{\cdot}\vec{x} + \frac{\dot{R}}{R}\dot{\vec{x}}$$

$$= \frac{\ddot{R}R - \dot{R}^2}{R^2}\vec{x} + \left(\frac{\dot{R}}{R}\right)^2\vec{x}$$

$$= \frac{\ddot{R}}{R}\vec{x} \tag{2.1.2}$$

If that corresponds to inertial, i.e. force-free motion, then Newton's equation will change to

$$\vec{F} = m\left(\ddot{\vec{x}} - \frac{\ddot{R}}{R}\vec{x}\right) \tag{2.1.3}$$

<u>expressing that "force" is the cause for deviations from inertial motion.</u>

Force free motion for

A) $R(t) \sim t^{2/3} \rightsquigarrow \frac{\ddot{R}}{R} = -\frac{2}{9}t^{-2}$ (2.1.4)

B) $R(t) \sim \exp(\lambda t) \rightsquigarrow \frac{\ddot{R}}{R} = \lambda^2$ (2.1.5)

Case A :

$$\ddot{\vec{X}} = -\frac{2}{9} t^{-2} \vec{X} \qquad (2.1.6)$$

Is solved by $\vec{X} = t^n \vec{n}$ for some fixed $\vec{n} \in \mathbb{R}^3$, iff

$$n(n-1) = -\frac{2}{9}$$

$$n^2 - n + \frac{2}{9} = 0$$

$$n_{1,2} = \frac{1}{2} \pm \left(\frac{1}{4} - \frac{2}{9}\right)^{1/2}$$

$$n_1 = \frac{1}{2} + \frac{1}{6} = \frac{8}{12} = \frac{2}{3} \qquad (2.1.6a)$$

$$n_2 = \frac{1}{2} - \frac{1}{6} = \frac{4}{12} = \frac{1}{3} \qquad (2.1.6b)$$

Since (2.1.6) is linear, the general solution is a linear combination of the two particular ones just found:

$$\vec{X}(t) = \left(a\, t^{1/3} + b\, t^{2/3}\right) \vec{n} \qquad (2.1.7)$$

Initial conditions fix a and b:
We choose $\vec{n} = \vec{e}_x$ and require

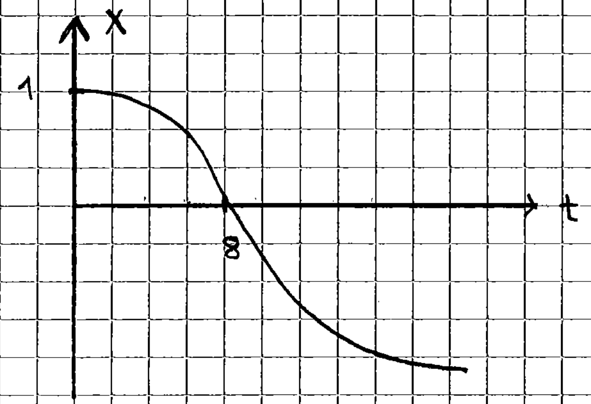
- $X(t=1) = r \Rightarrow a + b = r \qquad (2.1.8)$
- $\dot{X}(t=1) = 0 \Rightarrow \frac{a}{3} + \frac{2b}{3} = 0 \qquad (2.1.9)$

$$\Rightarrow a = 2r, \quad b = -r \qquad (2.1.10)$$

hence

$$X(t) = r\left(2t^{1/3} - t^{2/3}\right) \qquad (2.1.11)$$

This has a zero at $t = 8$



So the particle recedes and starts moving towards you, crossing the origin at $t = 8$ and then moving asymptotically with the Hubble flow, i.e. $X(t) \sim t^{2/3}$ in the <u>opposite direction</u>!. Note that the particle crosses the origin $X = 0$ at the time $t = 8$ independent of initial position r. This behaviour is typical for <u>decelerated expansion</u>. It is the deceleration that matters in the $\ddot{R}/R$ - term, i.e. $\ddot{R} < 0$, not the expansion $(\dot{R} > 0)$.

Case B:

$$\ddot{\vec{X}} = \lambda^2 \vec{X} \qquad (2.1.12)$$

$$\Rightarrow \vec{X}(t) = \left(a e^{\lambda t} + b e^{-\lambda t}\right) \vec{n} \qquad (2.1.13)$$

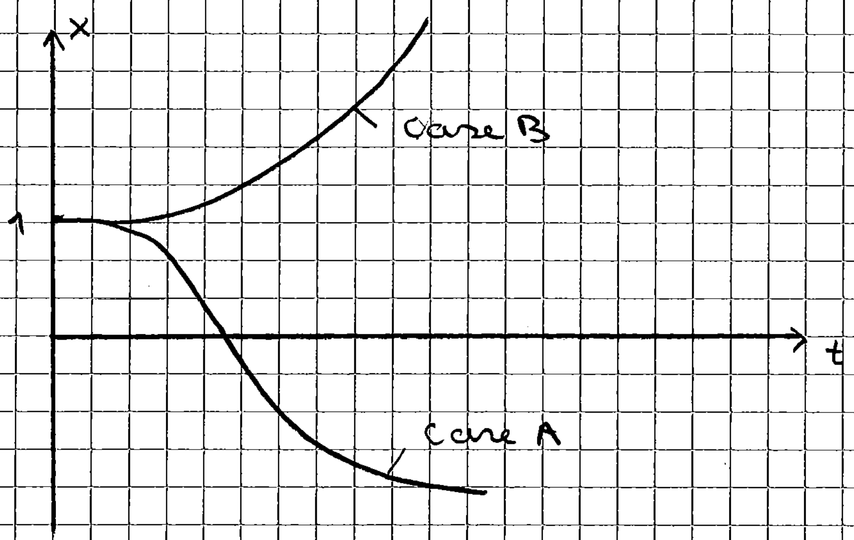
Again we choose $\vec{n} = \vec{e}_x$ and get from initial conditions

- $X(t=1) = r = a e^{\lambda} + b e^{-\lambda}$ (2.1.14)
- $\dot{X}(t=1) = 0 \Rightarrow \lambda a e^{\lambda} - \lambda b e^{-\lambda} = 0$ (2.1.15)

$$\Rightarrow a = \frac{r}{2} e^{-\lambda}, \quad b = \frac{r}{2} e^{\lambda} \qquad (2.1.16)$$

hence

$$X(t) = r \cosh\left(\lambda (t-1)\right) \qquad (2.1.17)$$



The particle recedes from observer, joining the exponential Hubble flow asymptotically for $t \to \infty$.

Problem 2

$$\vec{F} = -\vec{\nabla}(mV) = mc\,\vec{\nabla}\frac{1}{r} \qquad (2.2.1)$$

$$c > 0 \quad \text{(attractive force)}$$

$$\vec{F} = m\left(\ddot{\vec{x}} - \frac{\ddot{R}}{R}\vec{x}\right) \iff$$

$$mc\,\vec{\nabla}\frac{1}{r} = m\left(\ddot{\vec{x}} - \lambda^2\vec{x}\right) \iff$$

$$\ddot{\vec{x}} = c\,\vec{\nabla}(1/r) + \lambda^2\vec{x} \qquad (2.2.2)$$

An energy-conservation law is obtained by multiplication with $\dot{\vec{x}}$

$$\dot{\vec{x}}\cdot\ddot{\vec{x}} - c\,\dot{\vec{x}}\cdot\vec{\nabla}\frac{1}{r} - \lambda^2\dot{\vec{x}}\vec{x} = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\vec{x}}^2 - \frac{c}{r} - \frac{\lambda^2}{2}r^2\right) = 0$$

$$\frac{1}{2}\dot{\vec{x}}^2 - \frac{c}{r} - \frac{\lambda^2}{2}r^2 = \mathcal{E} = \text{const} \qquad (2.2.3)$$

($\mathcal{E}$ = total energy per mass)

Angular momentum conservation implies planar motion and

$$L = m r^2\dot{\varphi} = \text{const}$$

$$\curvearrowright \quad r^2\dot{\varphi} = l = L/m \qquad (2.2.4)$$

Then

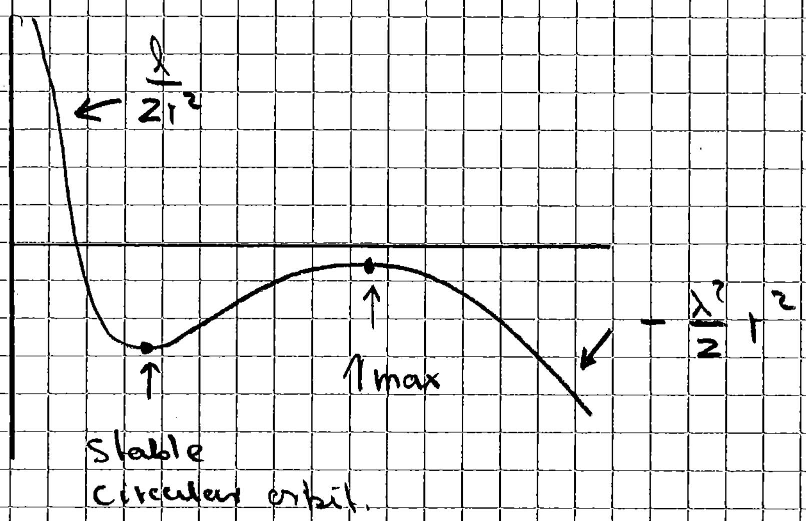
$$\dot{\vec{x}}^2 = \dot{r}^2 + r^2 \dot{\varphi}^2$$

$$= \dot{r}^2 + l^2/r^2 \qquad (2.2.5)$$

hence

$$\frac{1}{2}\dot{r}^2 + V(r) = \mathcal{E}$$

$$V(r) = \frac{l^2}{2r^2} - \frac{c}{r} - \frac{\lambda^2}{2} r^2 \qquad (2.2.6)$$



(2.2.7)

For $l = 0$ the attractive force/m is zero if

$$\left.\frac{d}{dr} V(r)\right|_{l=0} = \frac{c}{r^2} - \lambda^2 r = 0$$

or

$$r = \left(\frac{c}{\lambda^2}\right)^{1/3} =: r_c \qquad (2.2.8)$$

No circular orbit can be beyond that.

From $R(t) \sim \exp(\lambda t)$ have

$$q_0 = -\frac{\ddot{R}R}{\dot{R}^2} = -\frac{\ddot{R}}{R} H_0^{-2} \qquad (2.2.9)$$

i.e.
$$\lambda^2 = -q_0 H_0^2 \qquad (2.2.10)$$

and
$$H_0 \cong 70 \text{ km} \cdot s^{-1} \cdot Mpc^{-1} \qquad (2.2.11)$$

$$q_0 \cong -0.6 \qquad (2.2.12)$$

For the gravitational case we have

$$C = GM \qquad (2.2.13)$$

M = central mass. Then

$$r_c = \left(\frac{C}{\lambda^2}\right)^{1/3} = \left(\frac{GM}{-q_0 H_0^2}\right)^{1/3}$$

$$= \left(\frac{2GM/c^2}{-2q_0 \; H_0^2/c^2}\right)^{1/3}$$

$$= \left(\frac{-1}{2q_0}\right)^{1/3} \left(R_M R_H^2\right)^{1/3} \qquad (2.2.14)$$

Where

$$R_M = \frac{2GM}{c^2} = \text{grav. radius} \qquad (2.2.15)$$

$$R_H = H_0/c = \text{Hubble radius} \qquad (2.2.16)$$

Note that

$$\left(\frac{-1}{2q_0}\right)^{1/3} \cong \left(\frac{1}{1.2}\right)^{1/3} \cong 0.94$$

Also

$$R_M = R_\odot \frac{M}{M_\odot}$$

where $R_\odot = \frac{2GM_\odot}{c^2} \cong 3\,\mathrm{km}$

Then

$$\underline{\underline{r_c \cong \left(R_\odot \cdot R_H^2\right)^{1/3} \left(\frac{M}{M_\odot}\right)^{1/3}}}$$

Have

$$R_H = \frac{c}{H_0} = \frac{3 \cdot 10^5\ \mathrm{km} \cdot s^{-1}}{70 \cdot \mathrm{km} \cdot s^{-1}}\ \mathrm{Mpc}$$

$$= 4{,}3 \cdot 10^3\ \mathrm{Mpc} = 4.3\ \mathrm{Gpc}$$

$$= 4.3\ 10^9\ 3.26\ \mathrm{ly}$$

$$= 4.3 \cdot 10^9 \cdot 3.26 \cdot 9{,}5 \cdot 10^{12}\ \mathrm{km}$$

$$\cong 10^{23}\ \mathrm{km}$$

$$\Rightarrow \left(R_\odot \cdot R_H^2\right)^{1/3} \cong 3.7\ 10^{15}\ \mathrm{km}$$

$$= 398\ \mathrm{ly}$$

Hence we get for r_c

$$r_c \cong 398\,ly \left(\frac{M}{M_\odot}\right)^{1/3} \qquad (2.2.17)$$

This means that for a galaxy of 10^{12} solar masses the Hubble flow starts to destabilize bound orbits at a distance of

$$r_c^{(Gal)} = 398 \cdot 10^4\,ly$$
$$\cong 4\;10^6\,ly > Mpc \qquad (2.2.18)$$

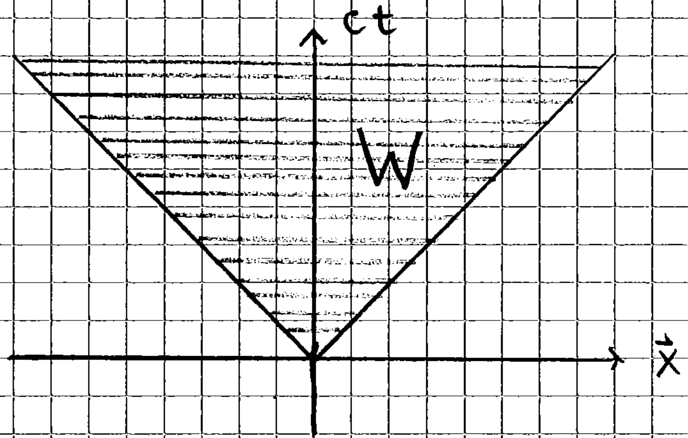
which is larger than the galaxy (10^5 ly) and of the order of galaxy distances in a group

$\Rightarrow$ Hubble flow starts to expand gravitationally bound systems from galaxy-cluster scales on.

Problem 3

$$W = \{ (ct, \vec{x}) \in \mathbb{R}^4 : \ ct > r \} \quad (2.3.1)$$

$$r = \|\vec{x}\|$$



(2.3.2)

"Radial" timelike vector field in W

$$U = c \, \frac{x^\alpha}{\sqrt{\eta(x,x)}} \, \partial_\alpha \quad (2.3.3)$$

Introduce coordinates $(\tau, \varrho, \theta, \varphi)$ via

$$ct = c\tau \cosh(\varrho) \quad (2.3.4)$$

$$\vec{x} = c\tau \sinh(\varrho) \, \vec{n} \quad (2.3.5)$$

$$\vec{n} = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix} \quad (2.3.6)$$

Have $\vec{n}^2 = 1$ hence

$$\vec{n} \cdot d\vec{n} = 0 \quad (2.3.7)$$

$$\eta = c^2 dt \otimes dt - d\vec{x} \otimes d\vec{x} \tag{2.3.8}$$

$$d(ct) = c\, d\tau \cosh(\varrho) + c\tau \sinh(\varrho)\, d\varrho \tag{2.3.9}$$

$$d\vec{x} = c\, d\tau \sinh(\varrho)\, \vec{n} + c\tau \cosh(\varrho)\, \vec{n}\, d\varrho + c\tau \sinh(\varrho)\, d\vec{n} \tag{2.3.10}$$

$\Rightarrow$

$$\begin{aligned}
\eta = {} & c^2 \cosh^2(\varrho)\, d\tau \otimes d\tau \\
& + c^2\tau^2 \sinh^2(\varrho)\, d\varrho \otimes d\varrho \\
& + c^2\tau \sinh(\varrho)\cosh(\varrho)(d\tau \otimes d\varrho + d\varrho \otimes d\tau) \\
& - c^2 \sinh^2(\varrho)\, d\tau \otimes d\tau \\
& - c^2\tau^2 \cosh^2(\varrho)\, d\varrho \otimes d\varrho \\
& - c^2\tau \sinh(\varrho)\cosh(\varrho)(d\tau \otimes d\varrho + d\varrho \otimes d\tau) \\
& - c^2\tau^2 \sinh^2(\varrho)\, d\vec{n} \otimes d\vec{n} \\
= {} & c^2 d\tau \otimes d\tau - c^2\tau^2 d\varrho \otimes d\varrho \\
& - c^2\tau^2 \sinh^2(\varrho)(d\theta \otimes d\theta + \sin^2\theta\, d\varphi \otimes d\varphi)
\end{aligned} \tag{2.3.11}$$

where we used (2.3.7) and

$$d\vec{n} \overset{\cdot}{\otimes} d\vec{n} = d\theta \otimes d\theta + \sin^2\theta \, d\varphi \otimes d\varphi \quad (2.3.12)$$

Hence have FLRW form of metric η

$$\eta = c\,d\tau \otimes c\,d\tau - a^2(\tau)\,\hat{g} \quad (2.3.13)$$

with

$$a(\tau) = c\tau \quad (2.3.14)$$

$$\hat{g} = d\rho \otimes d\rho + \sinh^2(\rho)\,(d\theta \otimes d\theta + \sin^2(\theta)\,d\varphi \otimes d\varphi). \quad (2.3.15)$$

= metric of constant curvature $k = -1$. ($\rightarrow$ hyperbolic geometry)

From (2.3.4-5) have

$$(ct)^2 - \|\vec{X}\|^2 = \eta_{\alpha\beta} X^\alpha X^\beta$$
$$= c^2 \tau^2 \quad (2.3.16)$$

$$\operatorname{artanh}\left(\frac{r}{ct}\right) = \rho \quad (2.3.17)$$

From (2.3.16)

$$\tau = \frac{1}{c} \left(\eta_{\alpha\beta} X^\alpha X^\beta\right)^{1/2}$$
$$= \frac{1}{c} \left(\eta(x,x)\right)^{1/2} \quad (2.3.18)$$

Hence

$$\partial_\alpha \tau = \frac{1}{c} \frac{\eta_{\alpha\beta} X^\beta}{(\eta(X,X))^{1/2}}$$

$$= \frac{1}{c} \frac{X_\alpha}{(\eta(X,X))^{1/2}} \qquad (2.3.19)$$

Also from (2.3.4-5), since the X^α depend linearly on τ,

$$\frac{\partial X^\alpha}{\partial \tau} = \frac{X^\alpha}{\tau} = c \frac{X^\alpha}{(\eta(X,X))^{1/2}} \qquad (2.3.20)$$

using (2.3.18) in the last step
Hence

$$\frac{\partial}{\partial \tau} = \frac{\partial X^\alpha}{\partial \tau} \frac{\partial}{\partial X^\alpha}$$

$$= c \frac{X^\alpha}{\sqrt{\eta(X,X)}} \partial_\alpha$$

$$= u \qquad (2.3.21)$$

In the new $(\tau, \xi, \theta, \varphi)$ coordinates the vector field u is just $\partial/\partial\tau$.

The divergence of u is

$$\nabla_\alpha u^\alpha = c\,\partial_\alpha\left(\frac{X^\alpha}{\sqrt{\eta(X,X)}}\right)$$

$$= c\left[\frac{4}{\sqrt{\eta(X,X)}} - \frac{X^\alpha \eta_{\alpha\beta} X^\beta}{(\eta(X,X))^{3/2}}\right]$$

$$= \frac{3c}{\sqrt{\eta(X,X)}} = \frac{3}{\tau} \qquad (2.3.18)$$

which diverges for $\tau \searrow 0$.
Is this a singularity? ("Big Bang")

The 3-dimensional manifolds τ = const. have constant sectional curvature

$$\frac{k}{a^2(\tau)} = \frac{-1}{c^2\tau^2} \qquad (2.3.19)$$

which diverges for $\tau \searrow 0$. The spacetime is, however flat!.
The Kretschmann scalar must be zero. Indeed, from Leture 4, formula (4.83) have

$$K = \frac{12}{a^4}\left[\left(\frac{\ddot{a}a}{c^4}\right)^2 + \left(k + \frac{\dot{a}^2}{c^2}\right)^2\right] \qquad (2.3.20)$$

Here $a(\tau) = c\tau$ and $k = -1$

hence $\ddot{a} = 0$ and $\dot{a}^2/c^2 = 1$,

so that $K = 0$.

Is this spacetime, therefore, to be considered singular? Well, not the spacetime, but the cosmological model.

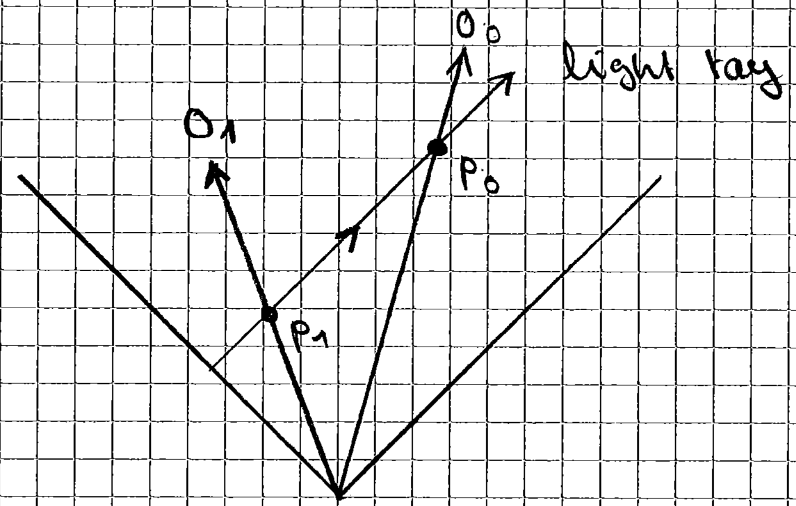
Problem 4

We know from Lecture 4, formula (4.30-31), which applies here for $k=-1$ and $\chi = \varrho$, that our metric is FLRW and also of the form (4.17) for $k=-1$. Hence it follows from the addendum to Lecture 4 that it is of constant curvature. We shall not repeat here the calculation in coordinates, e.g. $(\varrho, \theta, \varphi)$. The result is obviously

$$R_{abcd} = -(\hat{g}_{ac}\hat{g}_{bc} - \hat{g}_{ad}\hat{g}_{bc}) \qquad (2.4.1)$$

Each constant time τ-slice is, because $\eta(X,X) = c^2\tau^2$, acted upon by the Lorentz group, and since the metric the slices carry are that induced by the ambient $(\mathbb{R}^4, \eta)$, $SO(1,3)$ acts as isometries on each $\tau =$ const. slice

Problem



A light ray intersects the world lines of observers O_1 and O_0 at events P_1 and P_0, respectively

Cosmological redshift satisfies

$$z = \frac{\nu_1 - \nu_0}{\nu_0} = \frac{a(\tau_0)}{a(\tau_1)} - 1 \qquad (\quad)$$

$$= \frac{\tau_0}{\tau_1} - 1 \qquad (2.5.1)$$

Along a radially outgoing light ray we have

$$c \, d\tau = a(\tau) \, d\varrho \qquad (2.5.2)$$

or $\quad \ln(\tau_0 / \tau_1) = \varrho_0 - \varrho_1 = \varrho_0$

$$\rightsquigarrow \quad z = \exp(\varrho_0) - 1 \qquad (2.5.3)$$