Sheet 3 : Solutions

Problem 1

The Lagrange-function for the energy functional of the geodesic problem is

$$L = \frac{1}{2}\{c^2\dot{t}^2 - a^2(t)\dot{\chi}^2 - a^2(t)\Sigma_k^2(\chi)(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2)\} \tag{3.1.1}$$

The Euler-Lagrange equations are

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{z}^\alpha} - \frac{\partial L}{\partial z^\alpha} = 0 \tag{3.1.2}$$

where $z^\alpha(\lambda) = (t(\lambda), \chi(\lambda), \theta(\lambda), \varphi(\lambda))$

t)
$$\frac{\partial L}{\partial \dot{t}} = c^2\dot{t} \tag{3.1.3}$$

$$\frac{\partial L}{\partial t} = -aa'(\dot{\chi}^2 + \Sigma_k^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2)) \tag{3.1.4}$$

$$\Rightarrow\ c^2\ddot{t} + aa'(\dot{\chi}^2 + \Sigma_k^2(\chi)(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2)) = 0 \tag{3.1.5}$$

χ)
$$\frac{\partial L}{\partial \dot{\chi}} = -a^2\dot{\chi} \tag{3.1.6}$$

$$\frac{\partial L}{\partial \chi} = -a^2\Sigma_k\Sigma_k'(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) \tag{3.1.7}$$

$$(-a^2\dot{x})^{\cdot} + a^2\Sigma_k\Sigma_k'(\dot{\theta}^2+\sin^2\theta\,\dot{\varphi}^2)=0 \qquad (3.1.8)$$

$\theta)$

$$\frac{\partial L}{\partial\dot{\theta}} = -a^2\Sigma_k^2\,\dot{\theta} \qquad (3.1.9)$$

$$\frac{\partial L}{\partial\theta} = -a^2\Sigma_k^2\,\sin\theta\cos\theta\,\dot{\varphi}^2 \qquad (3.1.10)$$

$$\Rightarrow \left(-a^2\Sigma_k^2\,\dot{\theta}\right)^{\cdot} + a^2\Sigma_k^2\,\sin\theta\cos\theta\,\dot{\varphi}^2 = 0 \qquad (3.1.11)$$

$\varphi)$

$$\frac{\partial L}{\partial\dot{\varphi}} = -a^2\Sigma_k^2\,\sin^2\theta\,\dot{\varphi} \qquad (3.1.12)$$

$$\frac{\partial L}{\partial\varphi} = 0 \qquad (3.1.13)$$

$$\Rightarrow \left(-a^2\Sigma_k^2\,\sin^2\theta\,\dot{\varphi}\right)^{\cdot} = 0 \qquad (3.1.14)$$

From (3.1.11) we see that if initially $\theta(\lambda=\lambda_0)=\pi/2$ and $\dot{\theta}(\lambda=\lambda_0)=0$ (motion starts in equatorial plane) then $\theta(\lambda)=\pi/2$ for all λ, i.e. motion remains entirely in equatorial plane.

From (3.1.14) we see that if initially $\varphi(\lambda=\lambda_0)=0$ and $\dot{\varphi}(\lambda=\lambda_0)=0$ (motion starts radially) then

$\varphi(\lambda) = 0$ for all λ, i.e. motion remains radially. Hence we get solution $\theta(\lambda) = \pi/2$, $\varphi(\lambda) = 0$ with $\dot{\theta}(\lambda) = \dot{\varphi}(\lambda) = 0$.

For that solution (3.1.5) and (3.1.8) become

$$c^2 \ddot{t} + a a' \dot{\chi}^2 = 0 \tag{3.1.15}$$

$$a^2 \dot{\chi} = k = \text{const.} \tag{3.1.16}$$

In addition, since we consider timelike geodesics in affine parametrisation, using $\lambda = \tau =$ proper time, we have

$$g_{\alpha\beta}(z(\lambda))\, \dot{z}^\alpha(\lambda)\, \dot{z}^\beta(\lambda) = c^2 \tag{3.1.17}$$

which in our case leads to

$$c^2 \dot{t}^2 - a^2 \dot{\chi}^2 = c^2 \tag{3.1.18}$$

This reduces our problem of radial geodesic motion to (3.1.15, 16, 18).

In fact, solutions to (3.1.15, 16, 18) give the most general solution, for we can always choose our (χ, θ, φ) coordinates to be centred at initial position; i.e. $x=0$ corresponds to initial space point.

Now, equations (3.1.15, 16) imply (3.1.18): replace $\dot{x}$ in (3.1.15) by (3.1.16) and get

$$c^2\ddot{t} + aa' k^2/a^4 = 0$$

Multiplication by $\dot{t}$ gives

$$c^2\ddot{t}\dot{t} + k^2 a'\dot{t}/a^3 = 0$$

$$\Leftrightarrow \frac{1}{2}\left(c^2\dot{t}^2 - k^2/a^2\right)^{\cdot} = 0$$

$$\overset{(16)}{\Leftrightarrow} c^2\dot{t}^2 - a^2\dot{x}^2 = K = \text{const}$$

which upon choosing $K = c$ ($\lambda = \tau$) is (3.1.18).

Conversely, (3.1.16, 18) imply (3.1.15): Differentiation of (3.1.16, 18) give

$$\ddot{x}a^2 + 2aa'\dot{t}\dot{x} = 0$$

$$c^2\dot{t}\ddot{t} - aa'\dot{t}\dot{x}^2 - a^2\dot{x}\ddot{x} = 0$$

Replacing $a^2\ddot{x}$ in the second by first:

$$\dot{t}\left[c^2\ddot{t} + aa'\dot{x}^2\right] = 0$$

implying (3.1.15) for $\dot{t} \neq 0$ (time-like geodesics always have $\dot{t} \neq 0$.)

Hence the geodesic equations are equivalent to (3.1.16) + (3.1.18).
The integration would now proceed as follows: Replace $\dot{x}$ in (3.1.18) by (3.1.16)

$$c^2 \dot{t}^2 - a^2 \frac{k^2}{a^4} = c^2$$

$$\Rightarrow \frac{dt}{\left[1 + \left(\frac{k}{c\,a(t)}\right)^2\right]^{1/2}} = d\tau \qquad (3.1.19)$$

Which can be integrated for given $a(t)$. Then, given $t(\tau)$ from that, we have from (3.1.16)

$$dx = k \frac{d\tau}{a^2(t(\tau))} . \qquad (3.1.20)$$

The four-velocity of the particle moving along the geodesic is

$$u = \dot{t} \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} \qquad (3.1.21)$$

$$= c\dot{t}\, e_0 + a\dot{x}\, e_1 \qquad (3.1.22)$$

where $e_0 = \frac{1}{c}\frac{\partial}{\partial t}$, $e_1 = \frac{1}{a}\frac{\partial}{\partial x}$ (3.1.23)

are orthonormal.

Hence using

$$\dot{t} = \left[1 + \left(\frac{k}{ca}\right)^2\right]^{1/2} \qquad (3.1.24)$$

$$\dot{x} = \frac{k}{a^2} \qquad (3.1.25)$$

have

$$u = c\left[1 + \left(\frac{k}{ca}\right)^2\right]^{1/2} e_0 + \frac{k}{a} e_1 \qquad (3.1.26)$$

The four momentum is

$$p = m u \qquad (3.1.27)$$

where m = rest mass.

The ordinary (3 dimensional) momentum in the rest frame of the cosmological observer is that part of p perpendicular to $\frac{\partial}{\partial t}$, i.e. perpendicular to e_0, i.e from (3.1.26)

$$\vec{p} = (mk/a)\, e_1 \qquad (3.1.28)$$

$$\Rightarrow \quad a\,\|\vec{p}\| = mk = \text{const.} \qquad (3.1.29)$$

This does not violate "momentum conservation" since that does not hold in general spacetimes and also not in FLRW. Conserved along geodesics are scalar quantities like $g(\dot{z}, K)$ for each Killing vector K:

$$\dot{z}^\alpha \nabla_\alpha \dot{z}^\beta K_\beta$$

$$= \underbrace{(\dot{z}^\alpha \nabla_\alpha \dot{z}^\beta)}_{=0 \text{ for geodesics}} K_\beta + \dot{z}^\alpha \dot{z}^\beta \nabla_\alpha K_\beta$$

$$= \dot{z}^\alpha \dot{z}^\beta \frac{1}{2} \underbrace{(\nabla_\alpha K_\beta + \nabla_\beta K_\alpha)}_{=0 \text{ for } K \text{ Killing}}$$

$$= 0.$$

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Problem 2

$$T = (\varrho + p/c^2)\, u \otimes u - g^{-1} p \in ST^2_0 M \quad (3.2.1)$$

$$\nabla : ST^2_0 M \to ST^2_1 M \quad (3.2.2)$$

$$\nabla e_\alpha = \omega^\beta{}_\alpha \otimes e_\beta \quad (3.2.3)$$

$$\nabla \theta^\alpha = - \omega^\alpha{}_\beta \otimes \theta^\beta \quad (3.2.4)$$

Let $\{e_0, e_a\}$ be orthonormal basis s.t. $e_0 = u/c$

$$\rightsquigarrow T = (\varrho c^2 + p)\, e_0 \otimes e_0 - g^{-1} p \quad (3.2.5)$$

$$\nabla T = d(\varrho c^2 + p) \otimes e_0 \otimes e_0$$
$$+ (\varrho c^2 + p)(\nabla e_0 \otimes e_0 + e_0 \otimes \nabla e_0)$$
$$- g^{-1}\, dp$$

$$= (c^2\, d\varrho + dp) \otimes e_0 \otimes e_0$$
$$+ (\varrho c^2 + p)\, \omega^\lambda{}_0 \otimes (e_\lambda \otimes e_0 + e_0 \otimes e_\lambda)$$
$$- g^{-1}\, dp$$

Contraction of this tensor expression in the first (and only) covariant and first (of two) contravariant slot gives the divergence, denoted by $\nabla \cdot T$:

$$\nabla \cdot T = e_0 (c^2 \rho + p)\, e_0 + (\rho c^2 + p) \left[\omega^{\lambda}{}_{\lambda 0}\, e_0 + \omega^{\lambda}{}_{00}\, e_\lambda \right] - (dp)^{\#} \qquad (3.2.6)$$

where $(dp)^{\#} := g^{-1}(dp, \cdot)$ is the gradient vector field of p. With respect to basis $\{e_\alpha\}$ have

$$(dp)^{\#} = \left(e_\lambda(p)\, \theta^\lambda \right)^{\#} = e_\lambda(p)\, (\theta^\lambda)^{\#} = e_\lambda(p)\, \eta^{\lambda\alpha}\, e_\alpha \qquad (3.2.7)$$

For FLRW metric we calculated in Lecture 4, formula (4.61)

$$\omega^a{}_0 = \left(\frac{\dot{a}}{ca} \right) \theta^a = \left(\frac{a_{,0}}{a} \right) \theta^a \qquad (3.2.8)$$

where $\dot{} = \partial/\partial t$, $a_{,0} = e_0(a) = \partial a / \partial x^0$.

Hence

$$\omega^\lambda{}_{\lambda 0} = \omega^l{}_{l0}$$
$$= \omega^l{}_0(e_l)$$
$$= \frac{a_{,0}}{a}\,\theta^l(e_l) = 3\,\frac{a_{,0}}{a} \qquad (3.2.9)$$

$$\omega^\lambda{}_{00} = \omega^\lambda{}_0(e_0)$$
$$= \begin{cases} 0 \text{ for } \lambda = 0 \\ \omega^l{}_0(e_0) = \frac{a_{,0}}{a}\,\theta^l(e_0) \end{cases}$$
$$= 0 \qquad (3.2.10)$$

Inserting (3.2.9-10) as well as (3.2.7) into (3.2.6) gives

$$\nabla\cdot T = \left[e_0(\rho c^2+p) + \frac{3a_{,0}}{a}(\rho c^2+p)\right] e_0$$
$$- e_\lambda(p)\,\eta^{\lambda\alpha}\,e_\alpha$$
$$= \left[e_0(\rho c^2) + \frac{3a_{,0}}{a}(\rho c^2+p)\right] e_0$$
$$+ \sum_{a=1}^{3} e_a(p)\,e_a \qquad (3.2.11)$$

Since $\eta^{\lambda\alpha} = \mathrm{diag}(1,-1,-1,-1)$

The vanishing of $\nabla \cdot T$ is then equivalent to

$$e_0(\rho c^2) + \frac{3 a_{|0}}{a}(\rho c^2 + p) = 0 \qquad (3.2.12)$$

and

$$e_a(p) = 0 \qquad (3.2.13)$$

The first line is equivalent to

$$e_0(\rho a^3) + e_0(a^3)\, p/c^2 = 0 \qquad (3.2.14)$$

or

$$(\rho a^3)^{\bullet} + (a^3)^{\bullet}\, p/c^2 = 0 \qquad (3.2.15)$$

Note that $e_a(\rho) = 0$ is not implied by $\nabla \cdot T = 0$. In FLRW models this would be a consequence of the equations of state $p = w \rho c^2$ if we assumed such.

Problem 3

$$g = a^2 \tilde{g} \tag{3.3.1}$$

$$\Rightarrow \left.\begin{aligned} g_{\alpha\beta} &= a^2 \tilde{g}_{\alpha\beta} \\ g^{\alpha\beta} &= a^{-2} \tilde{g}^{\alpha\beta} \end{aligned}\right\} \tag{3.3.2}$$

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda}\left(-g_{\beta\gamma,\lambda} + g_{\lambda\beta,\gamma} + g_{\gamma\lambda,\beta}\right)$$

$$= \frac{1}{2}\tilde{g}^{\alpha\lambda}\left(-\tilde{g}_{\beta\gamma,\lambda} + \tilde{g}_{\lambda\beta,\gamma} + \tilde{g}_{\gamma\lambda,\beta}\right)$$

$$+ \tilde{g}^{\alpha\lambda}\left(-\tilde{g}_{\beta\gamma}\frac{a_{,\lambda}}{a} + \tilde{g}_{\lambda\beta}\frac{a_{,\gamma}}{a} + \tilde{g}_{\gamma\lambda}\frac{a_{,\beta}}{a}\right)$$

$$= \tilde{\Gamma}^\alpha{}_{\beta\gamma} + \left(-\tilde{g}_{\beta\gamma}\tilde{g}^{\alpha\lambda}\frac{a_{,\lambda}}{a} + \delta^\alpha_\beta \frac{a_{,\gamma}}{a} + \delta^\alpha_\gamma \frac{a_{,\beta}}{a}\right) \tag{3.3.3}$$

A lightlike geodesic satisfies

$$\ddot{z}^\alpha + \left(\Gamma^\alpha{}_{\beta\gamma} \circ z\right)\dot{z}^\beta \dot{z}^\gamma = 0 \tag{3.3.4}$$

$$\text{and} \quad \left(g_{\alpha\beta} \circ z\right)\dot{z}^\alpha \dot{z}^\beta = 0 \tag{3.3.5}$$

Hence also

$$\left(\tilde{g}_{\alpha\beta} \circ z\right)\dot{z}^\alpha \dot{z}^\beta = 0 \tag{3.3.6}$$

Inserting (3.3.3) into (3.3.4) using (3.3.6) gives

$$\ddot{z}^\alpha + (\tilde{\Gamma}^\alpha_{\beta\gamma} \circ z)\dot{z}^\beta \dot{z}^\gamma + 2\,\dot{z}^\alpha \frac{(a_{,\beta} \circ z)\,\dot{z}^\beta}{(a \circ z)} = 0 \tag{3.3.7}$$

But

$$(a \circ z)^{\cdot} = (a_{,\beta} \circ z)\,\dot{z}^\beta \tag{3.3.8}$$

hence

$$\ddot{z}^\alpha + (\tilde{\Gamma}^\alpha_{\beta\gamma} \circ z)\,\dot{z}^\beta \dot{z}^\gamma = -2\,\frac{(a \circ z)^{\cdot}}{(a \circ z)}\,\dot{z}^\alpha \tag{3.3.9}$$

Under reparametrisation

$$f : \mathbb{R} \supseteq I \to I' \subseteq \mathbb{R} \tag{3.3.10}$$

$$h := f^{-1} : \mathbb{R} \supseteq I' \to I \subseteq \mathbb{R} \tag{3.3.11}$$

with

$$y^\alpha := z^\alpha \circ f^{-1} \tag{3.3.12}$$

$$\rightsquigarrow \quad z^\alpha = y^\alpha \circ f \tag{3.3.14}$$

have

$$\dot{z}^\alpha = (\dot{y}^\alpha \circ f)\,\dot{f} \tag{3.3.15}$$

$$\ddot{z}^\alpha = (\ddot{y}^\alpha \circ f)\,\dot{f}^2 + (\dot{y}^\alpha \circ f)\,\ddot{f} \tag{3.3.16}$$

Inserting this into (3.3.9) gives

$$(\ddot{y}^\alpha \circ f)\,\dot{f}^2 + (\dot{y}^\alpha \circ f)\,\ddot{f}$$

$$+ (\tilde{\Gamma}^\alpha{}_{\beta\gamma} \circ y \circ f)(\dot{y}^\alpha \circ f)(\dot{y}^\beta \circ f)$$

$$= -2\,\frac{(a \circ y \circ f)^\bullet}{(a \circ y \circ f)}\,(\dot{y}^\alpha \circ f)\,\dot{f}$$

$$= -2\left[\left(\frac{(a \circ y)^\bullet}{a \circ y}\right) \circ f\right](\dot{y}^\alpha \circ f)\,\dot{f}^2 \qquad (3.3.17)$$

where we used

$$(a \circ y \circ f)^\bullet = [(a \circ y)^\bullet \circ f]\,\dot{f} \qquad (3.3.18)$$

Dividing by $\dot{f}^2$ and composing with $h = f^{-1}$ from right gives

$$\ddot{y}^\alpha + (\tilde{\Gamma}^\alpha{}_{\beta\gamma} \circ y)\,\dot{y}^\alpha\,\dot{y}^\beta$$

$$= -\left[\frac{\ddot{f}}{\dot{f}^2} \circ h + 2\,\frac{(a \circ y)^\bullet}{(a \circ y)}\right] \qquad (3.3.19)$$

From

$$h \circ f = \mathrm{id}_I \qquad (3.3.20)$$

$$\Rightarrow (\dot{h} \circ f)\,\dot{f} = \mathrm{id} \qquad (3.3.21)$$

$$\Rightarrow \frac{1}{\dot{f}} = \dot{h} \circ f \qquad (3.3.22)$$

$$\Rightarrow \quad -\frac{\ddot{\varphi}}{\dot{\varphi}^2} = \left(\frac{1}{\dot{\varphi}}\right)^{\cdot} = (\dot{h}\circ\varphi)^{\cdot}$$

$$= (\ddot{h}\circ\varphi)\,\dot{\varphi}$$

$$= \frac{\ddot{h}}{\dot{h}}\circ\varphi \tag{3.3.24}$$

Hence

$$-\frac{\ddot{\varphi}}{\dot{\varphi}^2}\circ h = \frac{\ddot{h}}{\dot{h}} \tag{3.3.25}$$

so that (3.3.19) becomes

$$\ddot{y}^\alpha + (\tilde{\Gamma}^\alpha_{\beta\gamma}\circ y)\,\dot{y}^\beta\dot{y}^\gamma$$

$$= \left[\frac{\ddot{h}}{\dot{h}} - 2\,\frac{(a\circ y)^{\cdot}}{(a\circ y)}\right]\dot{y}^\alpha \tag{3.3.26}$$

Problem 4

From the previous problem we see that y^i satisfies the geodesic eqn. for $\tilde{g}$ iff

$$\frac{\ddot{h}}{\dot{h}} = 2 \frac{(a \circ y)^{\cdot}}{a \circ y} \tag{3.4.1}$$

We only have to show that this is equivalent to

$$\frac{\ddot{f}}{\dot{f}} = -2 \frac{(a \circ z)^{\cdot}}{a \circ z} \tag{3.4.2}$$

But from (3.3.25) we know that (3.4.1) is equivalent to

$$\frac{\ddot{h}}{\dot{h}} = -\frac{\ddot{f}}{\dot{f}^2} \circ h = 2 \frac{(a \circ y)^{\cdot}}{(a \circ y)} \tag{3.4.3}$$

Composing with f from right, multiplying with $\dot{f}$, and using

$$\begin{aligned} ((a \circ y)^{\cdot} \circ f)\, \dot{f} &= (a \circ y \circ f)^{\cdot} \\ &= (a \circ z)^{\cdot} \end{aligned} \tag{3.4.4}$$

We get

$$-\frac{\ddot{\varphi}}{\dot{\varphi}} = 2 \frac{(a \circ z)^{\cdot}}{(a \circ z)} . \qquad (3.4.5)$$

(3.4.1) is equivalent to

$$[\ln \dot{h} - 2 \ln (a \circ y)]^{\cdot} = 0 \qquad (3.4.6)$$

$$\Leftrightarrow \quad \dot{h} \, (a \circ y)^{-2} = c' = \text{const.} \qquad (3.4.7)$$

Note $h : \mathbb{R}^3 \supseteq I' \to I \subseteq \mathbb{R}$

$$\lambda = h(\lambda')$$

$$\curvearrowright \quad \dot{h} = \frac{d\lambda}{d\lambda'} \qquad (3.4.8)$$

$$\Rightarrow \quad \frac{d\lambda}{d\lambda'} \left(a \circ y(\lambda')\right)^{-2} = c'$$

$$\Rightarrow \lambda = h(\lambda') = c' \int^{\lambda'} d\tilde{\lambda} \, \left(a \circ y(\tilde{\lambda})\right)^{2} \qquad (3.4.9)$$

Similarly, (3.4.5) is equivalent to

$$\ln \dot{\varphi} + 2 \ln (a \circ z)]^{\cdot} = 0 \qquad (3.4.10)$$

$$\Leftrightarrow \quad \dot{\varphi} \, (a \circ z)^{2} = c \qquad (3.4.11)$$

With

$$\varphi : \mathbb{R} \supseteq I \to I' \subseteq \mathbb{R}$$

$$\lambda' = \varphi(\lambda)$$

$$\wedge \quad \dot{\varphi} = d\lambda' / d\lambda \qquad (3.4.12)$$

$$\Rightarrow \quad \frac{d\lambda'}{d\lambda} \, (a \circ z(\lambda))^2 = c$$

$$\Rightarrow \quad \lambda' = \varphi(\lambda) = \int^{\lambda} \frac{d\tilde{\lambda}}{(a \circ z(\tilde{\lambda}))^2} \qquad (3.4.13)$$

Problem 5

$$g = dx^0 \otimes dx^0 - a^2(x^0)\,\hat{g} \qquad (3.5.1)$$

$$= a^2(x^0(\eta))\,[\underbrace{d\eta \otimes d\eta - \hat{g}}_{\tilde{g}}] \qquad (3.5.2)$$

FLRW-metric is conformally equivalent to static metric $\tilde{g}$ which is a simple product metric on $\mathbb{R}$ with a constant curvature metric on $\hat{M}$. Null geodesics on (M, g) and $(M, \tilde{g})$ are related by simple reparametrisations (see previous exercise). Hence in order to determine all null geodesics on (M, g) we can solve the easier problem on $(M, \tilde{g})$.

Energy-functional for geodesics on $(M, \tilde{g})$:

$$L = \frac{1}{2}\left[\dot{\eta}^2 - \dot{\chi}^2 - \Sigma_k^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2)\right] \qquad (3.5.3)$$

Euler Lagrange for χ:

$$\ddot{\eta} = 0 \Rightarrow \eta = a\tilde{\lambda} + b \qquad (3.5.4)$$

$\uparrow$ affin wrt. $\tilde{g}$

E3.20

For lightlike geodesics have

$$\dot{\eta}^2 - \hat{g}_{ab} \dot{z}^a \dot{z}^b = 0 \qquad (3.5.5)$$

$$\Rightarrow \quad d\eta = \pm \underbrace{\left(\hat{g}_{ab}\, dz^a \otimes dz^b\right)^{1/2}}_{d\hat{s} \,=\, \text{proper length wrt } \hat{g}} \qquad (3.5.6)$$

$$\Rightarrow \quad d\tilde{\lambda} \sim d\hat{s}\,. \qquad (3.5.7)$$

From

$$\lambda = h(\tilde{\lambda}) = \tilde{c} \int^{\tilde{\lambda}} d\tilde{\lambda}'\, a^2(y(\tilde{\lambda}'))$$

$$= \tilde{c}' \int^{\eta} d\eta\, a^2(\eta)$$

$$= \tilde{c}' \int^{x^0} \underbrace{\frac{d\eta}{dx^0}}_{\frac{1}{a}}\, a^2(\eta)\, dx^0$$

$$= \tilde{c}' \int^{x^0} a(\tilde{x}^0)\, d\tilde{x}^0 \qquad (3.5.8)$$

which gives a simple relation between the affine parameter λ for lightlike geodesics in (M, g) and x^0. Note: x^0/c is, in turn, the proper time along the world-lines of cosmological observers.