Sheet 4: Solutions

Problem 1

The derivation of Mattig's equation starts with formula (7.66):

$$d_{G}^{(0)}(z) = \frac{c}{H_0} \int_{\frac{1}{z+1}}^{1} \frac{d\mu}{\left[(1-\Omega)\mu^2 + \Omega\mu\right]^{1/2}} \qquad (4.1.1)$$

and in Lecture 7 we showed step by step how to proceed in case $(1-\Omega) < 0$.

Now we assume

$$(1-\Omega) = \Omega_k = -\frac{kc^2}{H_0^2 a_0^2} > 0 \qquad (4.1.2)$$

i.e. negative curvature. Hence

$$(1-\Omega)\mu^2 + \Omega\mu =$$

$$(1-\Omega)\left[\mu^2 + \frac{\Omega}{1-\Omega}\mu\right] =$$

$$(1-\Omega)\left[\left(\mu + \frac{\Omega}{2(1-\Omega)}\right)^2 - \left(\frac{\Omega}{2(1-\Omega)}\right)^2\right] \qquad (4.1.3)$$

$$\Rightarrow \quad d_{G}^{(0)}(z) = \frac{c}{H_0\sqrt{1-\Omega}} \times$$

$$\int_{\frac{1}{1+z}}^{1} \frac{d\mu}{\left[\left(\mu + \frac{\Omega}{2(1-\Omega)}\right)^2 - \left(\frac{\Omega}{2(1-\Omega)}\right)^2\right]^{1/2}} \qquad (4.1.4)$$

The prefactor is, using (4.1.2) and $k=-1$,

$$\frac{c}{H_0\sqrt{1-\Omega}} = a_0 \qquad (4.1.5)$$

Using the variable

$$y = u + \frac{\Omega}{2(1-\Omega)} \qquad (4.1.6)$$

(4.1.4) becomes

$$d_{c}^{(0)}(z) = a_0 \int_{\frac{1}{1+z}+\frac{\Omega}{2(1-\Omega)}}^{1+\frac{\Omega}{2(1-\Omega)}} \frac{dy}{\left[y^2 - \left(\frac{\Omega}{2(1-\Omega)}\right)^2\right]^{1/2}}$$

$$= a_0 \operatorname{arcosh}\left(\frac{2(1-\Omega)\,y}{\Omega}\right)\Bigg|_{\frac{1}{1+z}+\frac{\Omega}{2(1-\Omega)}}^{1+\frac{\Omega}{2(1-\Omega)}}$$

$$= a_0 \left\{ \operatorname{arcosh}\left(\frac{2-\Omega}{\Omega}\right) - \operatorname{arcosh}\left(\frac{2(1-\Omega)}{(1+z)\Omega}+1\right)\right\} \qquad (4.1.7)$$

The arcosh-functions obey

$$\operatorname{arcosh}(a) - \operatorname{arcosh}(b) = \operatorname{arsinh}\left(-a\sqrt{b^2-1} + b\sqrt{a^2-1}\right) \qquad (4.1.8)$$

This follows from

$$\sinh(\alpha-\beta) = \sinh(\alpha)\cosh(\beta) - \sinh(\beta)\cosh(\alpha) \qquad (4.1.9)$$

Applying arsinh and setting

$$\alpha = \operatorname{arcosh}(a), \quad \beta = \operatorname{arcosh}(b) \qquad (4.1.10)$$

this gives

$$\operatorname{arcosh}(a) - \operatorname{arcosh}(b) = \operatorname{arsinh}\left(b\sqrt{a^2-1} - a\sqrt{b^2-1} \right) \qquad (4.1.8)$$

In our case

$$a = \frac{2-\Omega}{\Omega}, \quad b = \frac{2(1-\Omega)}{(1+z)\Omega} + 1 \qquad (4.1.11)$$

$$\rightsquigarrow \quad b\,(a^2-1)^{1/2} - a\,(b^2-1)^{1/2}$$

$$= \frac{2-\Omega+z\Omega}{(1+z)\Omega}\left[\left(\frac{2-\Omega}{\Omega}\right)^2 - 1\right]^{1/2} - \frac{2-\Omega}{\Omega}\left[\left(\frac{(2-\Omega+z\Omega)^2}{(1+z)\Omega}\right)^2 - 1\right]^{1/2}$$

$$= \frac{2-\Omega+z\Omega}{(1+z)\Omega^2}\left[(2-\Omega)^2 - \Omega^2\right]^{1/2} - \frac{2-\Omega}{(1+z)\Omega^2}\left[(2-\Omega+z\Omega)^2 - (1+z)^2\Omega^2\right]^{1/2} \qquad (4.1.12)$$

The first quare-bracked is simply

$$\left((2-\Omega)^2-\Omega^2\right)^{1/2} = 2\,(1-\Omega)^{1/2} \qquad (4.1.13)$$

The second square bracket is

$$\left[(2-\Omega+z\Omega)^2-(1+z)^2\,\Omega^2\right]^{1/2}$$

$$=\Big[4+\cancel{\Omega^2}+\cancel{z^2\Omega^2}-4\Omega+4z\Omega-\underline{2z\Omega^2}$$
$$-\cancel{\Omega^2}-\cancel{z^2\Omega^2}-\underline{2z\Omega^2}\Big]^{1/2}$$

$$=\left[4z\Omega-4z\Omega^2+4(1-\Omega)\right]^{1/2}$$

$$=\left[4z\Omega(1-\Omega)+4(1-\Omega)\right]^{1/2}$$

$$=2\,(1-\Omega)^{1/2}\,(1+z\Omega)^{1/2} \qquad (4.1.14)$$

Hence

$$b\,(a^2-1)^{1/2}-a\,(b^2-1)^{1/2}$$

$$=2\,(1-\Omega)^{1/2}\left\{\frac{2-\Omega+z\Omega}{(1+z)\,\Omega^2}-\frac{(2-\Omega)}{(1+z)\,\Omega^2}\,(1+z\Omega)^{1/2}\right\}$$

$$\frac{2\,(1-\Omega)^{1/2}}{\Omega^2\,(1+z)}\left\{(\Omega-2)\left[(1+z\Omega)^{1/2}-1\right]+z\Omega\right\} \qquad (4.1.15)$$

Using again (4.1.5), i.e.

$$(1-\Omega)^{1/2}=\frac{c}{H_0\,a_0} \qquad (4.1.16)$$

this gives

$$b\,(a^2-1)^{1/2} - a\,(b^2-1)^{1/2}$$

$$= \frac{2c}{H_0 a_0} \frac{(\Omega-2)\left[(1+z\Omega)^{1/2}-1\right]+z\Omega}{\Omega^2(1+z)} \qquad (4.1.17)$$

exactly as in (7.78) for $k=+1$

Hence (4.1.7) becomes

$$d_G^{(0)}(z) = a_0 \operatorname{arsinh}^{-1} \Big\{$$

$$\frac{2c}{H_0 a_0} \frac{(\Omega-2)\left[(1+z\Omega)^{1/2}-1\right]+z\Omega}{\Omega^2(1+z)}$$

$$= a_0 \Sigma_k^{-1}\left(\frac{2c}{H_0 a_0} \frac{(\Omega-2)\left[(1+z\Omega)^{1/2}-1\right]+z\Omega}{\Omega^2(1+z)}\right) \qquad (4.1.18)$$

which is identical to (7.79).

Hence (7.57) gives for d_L and d_A the same expressions as (7.80):

$$d_L(z) = \frac{2c}{H_0} \frac{(\Omega-2)\left[(1+z\Omega)^{1/2}-1\right]+z\Omega}{\Omega^2} \qquad (4.1.19a)$$

$$d_A(z) = \frac{2c}{H_0} \frac{(\Omega-2)\left[(1+z\Omega)^{1/2}-1\right]+z\Omega}{(1+z)^2\,\Omega^2} \qquad (4.1.19b)$$

Finally, if $\Omega = 1$, (4.1.1) simply reads:

$$d_G^{(0)}(z) = \frac{c}{H_0} \int_{\frac{1}{z+1}}^{1} \frac{du}{u^{1/2}}$$

$$= \frac{2c}{H_0} u^{1/2} \Big|_{\frac{1}{1+z}}^{1}$$

$$= \frac{2c}{H_0} \left(1 - \frac{1}{\sqrt{1+z}} \right)$$

$$= \frac{2c}{H_0} \frac{(1+z) - \sqrt{1+z}}{(1+z)}$$

$$= a_0 \Sigma_k^{-1} \frac{2c}{H_0 a_0} \frac{(\Omega - 2)\left[(1+z\Omega)^{1/2} - 1\right] + z\Omega}{\Omega^2 (1+z)} \Bigg|_{\substack{\Omega = 1 \\ k = 0}}$$

Hence for $\Omega = 1$ and $k = 0$ we get again (4.1.18) and (4.1.19), which now read

$$d_L(z) = \frac{2c}{H_0} \left[1 + z - (1+z)^{1/2} \right]$$

$$d_A(z) = \frac{2c}{H_0} \left[\frac{1}{1+z} - \frac{1}{(1+z)^{3/2}} \right]$$

as already anticipated in (7.44) and (7.45), respectively.

All this proves the general validity of (7.80) in case $\Omega_{rad} = \Omega_\Lambda = 0$.

Problem 2

The expression

$$\frac{(\Omega-2)\left[(1+z\Omega)^{1/2}-1\right]+z\Omega}{\Omega^2(1+z)} \qquad (4.2.1)$$

has vanishing numerator and denominator for $\Omega \to 0$. Expanding

$$(1+z\Omega)^{1/2} = 1+\frac{z}{2}\Omega-\frac{1}{8}z^2\Omega^2+\cdots \qquad (4.2.2)$$

this fraction becomes

$$\frac{(\Omega-2)\left[\frac{1}{2}z\Omega-\frac{1}{8}z^2\Omega^2+\cdots\right]+z\Omega}{\Omega^2(1+z)}$$

$$= \frac{\frac{1}{2}z\Omega^2+\frac{1}{4}z^2\Omega^2+\mathcal{O}(\Omega^3)}{\Omega^2(1+z)}$$

$$= \underbrace{\frac{1}{4}\,\frac{z(z+2)}{1+z}}_{\text{limit for } \Omega \to 0} + \mathcal{O}(\Omega) \qquad (4.2.3)$$

But we can give the fraction another form that is often used in expressions for $d_G^{(0)}(z)$, $d_L(z)$, and $d_A(z)$.

For that we multiply the numerator and denominator by

$$(1+Z\Omega)^{1/2}+1 \;+\tfrac{1}{2}Z\Omega\,. \tag{4.2.4}$$

For the numerator we then get

$$\Big\{(\Omega-2)\big[(1+Z\Omega)^{1/2}-1\big]+Z\Omega\Big\}\Big\{(1+Z\Omega)^{1/2}+1+\tfrac{Z}{2}\Omega\Big\}$$

$$=(\Omega-2)\big[(1+Z\Omega)^{1/2}-1\big]\big[(1+Z\Omega)^{1/2}+1+\tfrac{Z}{2}\Omega\big]$$

$$+\,Z\Omega\,(1+Z\Omega)^{1/2}+Z\Omega+Z^2\Omega^2/2$$

$$=\;Z\Omega\,(\Omega-2)+\tfrac{Z}{2}\Omega\,(\Omega-2)\big[(1+Z\Omega)^{1/2}-1\big]$$

$$+\,Z\Omega\,(1+Z\Omega)^{1/2}+Z\Omega+Z^2\Omega^2/2$$

$$=\;Z\Omega^2-\cancel{2Z\Omega}^{\,1}+\tfrac{1}{2}Z\Omega^2\big[(1+Z\Omega)^{1/2}-1\big]$$

$$-\,Z\Omega\big[(1+\cancel{Z}^{\,2}\Omega)^{1/2}\cancel{-1}^{\,1}\big]+Z\Omega\,\cancel{(1+Z\Omega)}^{\,2}{}^{1/2}$$

$$+\,\cancel{Z\Omega}^{\,1}+Z^2\Omega^2/2$$

$$=\;\frac{Z\Omega^2}{2}\big[(1+Z\Omega)^{1/2}+1\;+Z\big] \tag{4.2.5}$$

For the denominator we get

$$\Omega^2\,(1+Z)\,\big(1+(1+Z\Omega)^{1/2}+\tfrac{1}{2}Z\Omega\big)\,. \tag{4.2.6}$$

Hence

$$\frac{(\Omega-2)\left[(1+z\Omega)^{1/2}-1\right]+z\Omega}{\Omega^2(1+z)}$$

$$= \frac{1}{2} \cdot \frac{z}{1+z} \cdot \frac{1+(1+z\Omega)^{1/2}+z}{1+(1+z\Omega)^{1/2}+\frac{1}{2}z\Omega} \qquad (4.2.7)$$

The corresponding expressions for $d_G^{(0)}$, d_L, and d_A are

$$d_G^{(0)}(z) = a_0 \Sigma_k^{-1}\left(\frac{c}{H_0 a_0} \frac{z}{1+z} \frac{1+(1+z\Omega)^{1/2}+z}{1+(1+z\Omega)^{1/2}+\frac{1}{2}z\Omega}\right) \qquad (4.2.8)$$

$$d_L(z) = \frac{c}{H_0} \cdot z \cdot \frac{1+(1+z\Omega)^{1/2}+z}{1+(1+z\Omega)^{1/2}+\frac{1}{2}z\Omega} \qquad (4.2.9a)$$

$$d_A(z) = \frac{c}{H_0} \frac{z}{(1+z)^2} \frac{1+(1+z\Omega)^{1/2}+z}{1+(1+z\Omega)^{1/2}+\frac{1}{2}z\Omega} \qquad (4.2.9b)$$

Problem 3

$$\Delta\varphi(D,z) = \frac{D}{d_A(z)} \qquad (4.3.1)$$

We have

$$d_A(z) = \frac{d_L(z)}{(1+z)^2} \qquad (4.3.2)$$

Hence

$$\Delta\varphi(D,z) = \frac{D}{\left(\frac{2c}{H_0}\right)} \frac{(1+z)^2\,\Omega^2}{(\Omega-2)\left[(1+z\Omega)^{1/2}-1\right]+z\Omega}$$

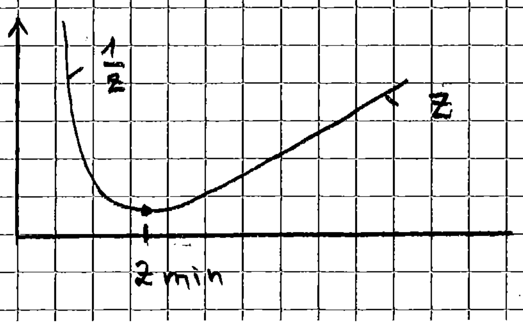
$$= \frac{D H_0}{c} \cdot \frac{1+z^2}{z} \frac{1+(1+z\Omega)^{1/2}+\frac{1}{2}\Omega z}{1+(1+z\Omega)^{1/2}+z} \qquad (4.3.3)$$

For small z this is to leading order

$$\Delta\varphi(D,z) \overset{(1)}{=} \frac{1}{z} D \frac{H_0}{c} \qquad (4.3.4)$$

For $z \to \infty$, on the other hand, this is

$$\Delta\varphi(D,z) \overset{(\infty)}{=} z \cdot \frac{\Omega}{2} \cdot D \cdot \frac{H_0}{c} \qquad (4.3.5)$$



E4.11

If $k = 0 \Rightarrow \Omega_k = 0 \Rightarrow \Omega = 1$, we have

$$\Delta\varphi(D,z) = D\left[d_L(z)/(1+z)^2\right]^{-1} \qquad (4.3.6)$$

$$\frac{d_L(z)}{(1+z)^2} = \frac{2c}{H_0}\,\frac{z-\left((1+z)^{1/2}-1\right)}{(1+z)^2}$$

$$= \frac{2c}{H_0}\left(\frac{1}{1+z}-\left(\frac{1}{1+z}\right)^{3/2}\right) \qquad (4.3.7)$$

This function is zero for $z = 0$ and zero for $z \to \infty$ and otherwise positive. Hence it must have a maximum where

$$\left[\frac{1}{1+z}-\left(\frac{1}{1+z}\right)^{3/2}\right]' = -\left(\frac{1}{1+z}\right)^2+\frac{3}{2}\left(\frac{1}{1+z}\right)^{5/2}$$

$$= \left(\frac{1}{1+z}\right)^{5/2}\left[\frac{3}{2}-\sqrt{1+z}\right] = 0$$

$$\Rightarrow \frac{9}{4} = 1+z \Leftrightarrow z = \frac{5}{4}. \qquad (4.3.8)$$

Hence $\Delta(D,z)$ has a minimum at $z = 5/4$. An object of true size D appears larger and larger if it recedes beyond $z = 5/4$.

[Compare discussion in Lecture 7, Fig. (7.49)]

Problem 4

$$\left.\begin{aligned} \Omega_{dust} &= 0.27 \\ \Omega_{rad} &= 8.25 \times 10^{-5} \\ \Omega_{\Lambda} &= 0.73 \\ \Omega_{k} &= 0 \end{aligned}\right\} \text{some quoted values. Not most recent} \qquad (4.4.1)$$

Have

$$\Omega_{dust} = \frac{8\pi G \rho_0^{dust}}{3 H_0^2} \qquad (4.4.2)$$

$$\Omega_{rad} = \frac{8\pi G \rho_0^{rad}}{3 H_0^2} \qquad (4.4.3)$$

$$\Rightarrow \frac{\Omega_{dust}}{\Omega_{rad}} = \frac{\rho_0^{dust}}{\rho_0^{rad}} = \frac{0.27}{8.25 \cdot 10^{-5}}$$

$$= 3.27 \cdot 10^{3} \qquad (4.4.4)$$

But

$$\left.\begin{aligned} \rho_0^{dust}\, a_0^3 &= \rho^{dust}(t)\, a^3(t) \\ \rho_0^{rad}\, a_0^4 &= \rho^{rad}(t)\, a^4(t) \end{aligned}\right\} \qquad (4.4.5)$$

or
$$\left.\begin{aligned} \rho^{dust}(t) &= (a_0/a(t))^3 \rho_0^{dust} = (1+z)^3 \rho_0^{dust} \\ \rho^{rad}(t) &= (a_0/a(t))^4 \rho_0^{rad} = (1+z)^4 \rho_0^{rad} \end{aligned}\right\} \qquad (4.4.6)$$

Hence

$$\frac{\rho^{dust}(t)}{\rho^{rad}(t)} = \left(\frac{1}{1+z}\right) \frac{\rho_0^{dust}}{\rho_0^{rad}}$$

$$= \frac{\Omega_{dust} / \Omega_{rad}}{1+z} \qquad (4.4.7)$$

Hence radiation dominance starts at redshifts above

$$(1+z) > \Omega_{dust} / \Omega_{rad}$$

$$= 3.27 \cdot 10^3 \qquad (4.4.8)$$

Other quoted values are (Caltech)

$$h_0 = 0.73 \pm 0.03$$

$$\Omega_m h_0^2 = 0.134 \pm 0.006 \qquad (4.4.9)$$

$$\Omega_{rad} h_0^2 = 2.47 \times 10^{-5}$$

$$\Rightarrow \frac{\Omega_m}{\Omega_{rad}} = \frac{0.134}{2.47} 10^5$$

$$= 5425 \qquad (4.4.10)$$

In any case, we observe

$$z_{rad\text{-}dom} \gg z_{recomb.} \qquad (4.4.11)$$

Problem 5

$$\left(\frac{dx}{d\lambda}\right)^2 + V(x) = E \qquad (4.5.1)$$

$$x = \frac{a}{a_0}, \quad \lambda = H_0 t, \quad E = \Omega_k \qquad (4.5.2)$$

$$V(x) = -\left(\frac{\Omega_{rad}}{x^2} + \frac{\Omega_{dust}}{x} + \Omega_\Lambda x^2\right) \qquad (4.5.3)$$

For $\Omega_k = \Omega_{rad} = 0$ this gives rise to

$$\frac{dx}{\left[\frac{\Omega_m}{x} + \Omega_\Lambda x^2\right]^{1/2}} = d\lambda \qquad (4.5.4)$$

We set

$$x = y^\alpha \qquad (4.5.5)$$

$$dx = \alpha y^{\alpha-1} dy \qquad (4.5.6)$$

Then the left-hand side becomes

$$\alpha \frac{dy}{\left[y^{2(1-\alpha)-\alpha}\Omega_m + y^{2(1-\alpha)+2\alpha}\Omega_\Lambda\right]^{1/2}} \qquad (4.5.7)$$

If we choose α such that the first exponent becomes zero i.e. $2-3\alpha=0$ $\to \alpha = 2/3$, then the second exponent is $2-2\alpha+2\alpha = 3\alpha = 2$.

Hence we set

$$X = y^{2/3} \tag{4.5.8}$$

and get

$$\int d\lambda = \frac{2}{3} \int \frac{dy}{\left[\Omega_m + \Omega_\Lambda y^2\right]^{1/2}}$$

$$= \frac{2}{3\sqrt{\Omega_m}} \int \frac{dy}{\left[1 + y^2 \frac{\Omega_\Lambda}{\Omega_m}\right]^{1/2}} \tag{4.5.9}$$

The following steps depend on the sign of the coefficient of the highest power of X, i.e. on the sign of Ω_Λ:

1. Case: $\Omega_\Lambda > 0$

Then we set

$$z = y \left(\frac{\Omega_\Lambda}{\Omega_m}\right)^{1/2} \tag{4.5.10}$$

$$\Rightarrow \int d\lambda = \frac{2}{3\sqrt{\Omega_\Lambda}} \int \frac{dz}{\left(1 + z^2\right)^{1/2}}$$

$$\Rightarrow \lambda - \lambda_0 = \frac{2}{3}\left(\Omega_\Lambda\right)^{-1/2} \left(\operatorname{arcsinh}(z) - \operatorname{arcsinh}(z_0)\right) \tag{4.5.11}$$

Choose w.l.o.g. $\lambda_0 = Z_0 = 0$, so that $Z = 0$ at $\lambda = 0$. Then

$$Z = \sinh\left(\frac{3}{2}\Omega_\Lambda^{1/2}\lambda\right) \qquad (4.5.12)$$

or

$$y = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/2} \sinh\left(\frac{3}{2}\Omega_\Lambda^{1/2}\lambda\right) \qquad (4.5.13)$$

or $$X(t) = \left\{\left(\frac{\Omega_m}{\Omega_\Lambda}\right) \sinh^2\left(\frac{3}{2}\Omega_\Lambda^{1/2}H_0 t\right)\right\}^{1/3}. \qquad (4.5.14)$$

using $\lambda = H_0 t$. Sometimes this is written in terms of cosh, using $\sinh^2(\alpha) = \frac{1}{2}(\cosh(2\alpha) - 1)$, as

$$X(t) = \left\{\left(\frac{\Omega_m}{2\Omega_\Lambda}\right)\cdot\left(\cosh\left(\frac{3}{2}\Omega_\Lambda^{1/2}H_0 t\right) - 1\right)\right\}^{1/3} \qquad (4.5.15)$$

This is a Big-Bang model expanding forever with singularity at $t = 0$ and current age at t_0, where

$$t = t_0 \iff a(t_0) = a_0 \iff X(t_0) = 1 \qquad (4.5.16)$$

so that

$$1 = \frac{\Omega_m}{\Omega_\Lambda} \sinh^2\left(\frac{3}{2}\Omega_\Lambda^{1/2}H_0 t_0\right) \qquad (4.5.17)$$

or

$$\boxed{t_0 = \frac{1}{H_0} \cdot \frac{2}{3} \cdot \Omega_\Lambda^{-1/2} \cdot \operatorname{arcsinh}\left(\frac{\Omega_\Lambda}{\Omega_m}\right)^{1/2}} \qquad (4.5.18)$$

"Age-of-the-Universe-Formula"
for $\Omega_k = \Omega_{rad} = 0$, $\Omega_\Lambda > 0$

Setting the values from Problem 4:

$$\Omega_\Lambda = 0.73 , \quad \Omega_m = 0.27 \qquad (4.5.19)$$

We get $\operatorname{arcsinh}(\Omega_\Lambda / \Omega_m) = 1.272$ and

$$\boxed{t_0 = \frac{1}{H_0} \times 0.9927 \cong H_0^{-1}} \qquad (4.5.20)$$

Using

$$H_0 = h \cdot 100 \cdot \frac{km/s}{Mpc} \qquad (4.5.21)$$

and $\quad 1\,Mpc = 3.086 \times 10^{19}\,km \qquad (4.5.22)$

$$\Rightarrow \frac{1}{H_0} = \frac{1}{h} \cdot 3.086 \cdot 10^{17}\,s$$

$$= \frac{1}{h} \cdot \frac{3.086 \cdot 10^{17}}{3.154 \cdot 10^{7}} \text{ years}$$

$$= \frac{1}{h} \cdot 9{,}786 \; 10^{9} \text{ years} \qquad (4.5.23)$$

Currently there are two "best-values" for h, one from observations of the Hubble-plot with SN 1A - standard candles, resulting in (Project SHOES)

$$\boxed{\begin{array}{l} h^{(SN)} = 0.7403 \pm 0.0142 \\ \text{(Adam Riess et al. (2019)} \end{array}} \qquad (4.5.24)$$

the other being measured on the Cosmic Microwave Backround (CMP) with the PLANCK-Satellite of ESA, resulting in

$$\boxed{\begin{array}{l} h^{(CMB)} = 0.677 \pm 0.004 \\ \text{(Akrami et al. 2018)} \end{array}} \qquad (4.5.25)$$

The resulting "ages of the universe" are

$$\boxed{\begin{array}{l} t_0^{(SN)} = 13.22 \times 10^9 \text{ years} \\ t_0^{(CMB)} = 14.5 \times 10^9 \text{ years} \end{array}} \qquad (4.5.26)$$

Compare recent (11/2020) issue of "Sterne and Weltraum".

Now we turn to the less relevant

2. Case: $\Omega_\Lambda < 0$

We then set instead of (4.5.10)

$$z = y\left(\frac{-\Omega_\Lambda}{\Omega_m}\right)^{1/2} \qquad (4.5.27)$$

$$\Rightarrow \int d\lambda = \frac{2}{3}\frac{1}{\sqrt{-\Omega_\Lambda}}\int \frac{dz}{(1-z^2)^{1/2}}$$

$$\Rightarrow \lambda - \lambda_0 = \frac{2}{3}(-\Omega_\Lambda)^{-1/2}\left(\arcsin(z) - \arcsin(z_0)\right) \qquad (4.5.28)$$

resulting in

$$X(t) = \left\{\frac{\Omega_m}{|\Omega_\Lambda|}\sin^2\left(\frac{3}{2}|\Omega_\Lambda|^{1/2}H_0 t\right)\right\}^{1/3} \qquad (4.5.29)$$

or, using $\sin^2(\alpha) = \frac{1}{2}(1-\cos(2\alpha))$,

$$X(t) = \frac{\Omega_m}{2|\Omega_\Lambda|}\left(1-\cos\left(3|\Omega_\Lambda|^{1/2}H_0 t\right)\right\}^{1/3} \qquad (4.5.30)$$

The corresponding formula for the "age of the universe" to (4.5.18) is

$$t_0 = \frac{1}{H_0}\cdot\frac{2}{3}\cdot|\Omega_\Lambda|^{-1/2}\arcsin\left(\frac{-\Omega_\Lambda}{\Omega_m}\right)^{1/2} \qquad (4.5.31)$$

or, since

$$\Omega_\Lambda = 1 - \Omega_m \qquad (4.5.32)$$

and $\Omega_m > 1$,

$$t_0 = \frac{1}{H_0} \cdot \frac{2}{3} (\Omega_m - 1)^{-1/2}$$

$$\times \operatorname{arsin} \left(1 - \frac{1}{\Omega_m}\right)^{1/2} \qquad (4.5.33)$$

If we set

$$\Omega_m = 1 + \varepsilon \qquad (4.5.34)$$

this becomes

$$t_0 = \frac{1}{H_0} \frac{2}{3} \quad \frac{\operatorname{arsin} (\varepsilon / 1+\varepsilon)^{1/2}}{\varepsilon^{1/2}}$$

$$= \frac{1}{H_0} \frac{2}{3} \left(1 - \frac{\varepsilon}{3} + \frac{\varepsilon^2}{5} - \dots\right) \qquad (4.5.35)$$

i.e for $\varepsilon \searrow 0$ we get

$$t_0^{(\Lambda = 0)} = \frac{2}{3} \frac{1}{H_0} \qquad (4.5.36)$$

Finally, the intermediate

3. Case: $\Omega_\Lambda = 0$, $\Omega_m = 1$

Then (4.5.4) just gives

$$dX \; X^{1/2} = d\lambda$$

$$\Rightarrow \frac{2}{3} X^{3/2} = \lambda \qquad (4.5.37)$$

(with $X(\lambda=0)=0$ as initial condition)

$$\Rightarrow X(t) = \left(\frac{3}{2} H_0 t\right)^{2/3} \qquad (4.5.38)$$

"matter dominated"

Again we see that $X(t_0) = 1$ for

$$t_0^{(\Lambda=0)} = \frac{2}{3} \frac{1}{H_0} \qquad (4.5.39)$$

as already seen in (4.5.36).

This can be contrasted to a "radiation dominated universe" in which $\Omega_{rad} = 1$ and all other Ω's equal to zero. In that case

We would have

$$\frac{dX}{\left[\frac{\Omega_{rad}}{X^2}\right]^{1/2}} = d\lambda \qquad (4.5.40)$$

$$\parallel$$

$$dX\, X$$

$$\Rightarrow \quad \frac{1}{2} X^2 = \lambda$$

$$X(t) = (2 H_0 t)^{1/2} \qquad (4.5.41)$$

$$\Rightarrow \quad t_0^{(RD)} = \frac{1}{2H_0} \qquad (4.5.42)$$

This may be a good approximation for very early stages $z \gg 10^5$. Note that (MD = matter (= dust) dominated RD = radiation dominated)

$$\frac{X_{MD}(t)}{X_{RD}(t)} \sim t^{2/3 - 1/2} = t^{1/6} \qquad (4.5.43)$$

which tends to zero for $t \searrow 0$ so that the phase of radiation dominance the singularity is approached slower in time as would be the case in a pure matter domination (lack of pressure)