Sheet 5 : Solutions

Problem 1

At time $t = t_0$ = "now" we have

$$T_0 = 2.72548 \pm 0.00057 \text{ K} \tag{5.1.1}$$

At time $t = t_1$ from which light to us is redshifted by z, the Temperature is

$$T_1 = (1+z)\, T_0 \tag{5.1.2}$$

so
$$z = \frac{T_1}{T_0} - 1 \tag{5.1.3}$$

T_0 we know and T_1 is calculated from Planck law through the condition that at $T = T_1$ maximum of Planck distribution occurs at frequency ν_{max} so that $h\nu_{max}$ = ionisation energy of hydrogen:

$$h\nu_{max} = 13.6 \text{ eV}. \tag{5.1.4}$$

The Planck law will translate that into a temperature T_1.

Planck-law (Lecture 8, (8.34)):

$$\varepsilon_\nu = h\nu \cdot \frac{8\pi}{c^3} \frac{\nu^2}{\exp(h\nu/kT)-1}$$

$$= \frac{8\pi}{c^3} h \frac{\nu^3}{\exp(h\nu/kT)-1}$$

$$= \frac{8\pi}{c^3} h \cdot \left(\frac{kT}{h}\right)^3 \frac{y^3}{\exp(y)-1} \quad (5.1.5)$$

where $y := \dfrac{h\nu}{kT}$ (5.1.6)

In order to find the extrema of ε_ν we need to find the extrema of

$$f(y) := \frac{y^3}{\exp(y)-1} \quad (5.1.7)$$

$$f'(y) = \frac{3y^2}{\exp(y)-1} - \frac{y^3 \exp(y)}{[\exp(y)-1]^2}$$

$$= \frac{3y^2(\exp(y)-1) - y^3 \exp(y)}{[\exp(y)-1]^2}$$

$$= \frac{y^2}{[\exp(y)-1]^2} \left[\exp(y)(3-y)-3\right] \quad (5.1.8)$$

Hence

$$f'(y) = 0 \iff$$

$$(3-y)\exp(y) = 3$$

$$(y-3)\exp(y-3) = -\frac{3}{e^3} \qquad (5.1.9)$$

(since $\exp(-3) = e^{-3}$).

Note that $f(0) = 0$ and $f(y \to \infty) = 0$ and $f(y) \geq 0$ for $y \geq 0$. Hence there must be a positive maximum of f for some $y > 0$.

(5.1.9) is satisfied for $y = 0$, but this not the maximum we are interested in. In fact, from (5.1.8)

$$f''(y) = \left[\frac{y^2}{[\exp(y)-1]^2}\right]' \left(\exp(y)(3-y)-3\right)$$

$$+ \frac{y^2}{[\exp(y)-1]^2}\Big[\exp(y)(3-y)-3$$

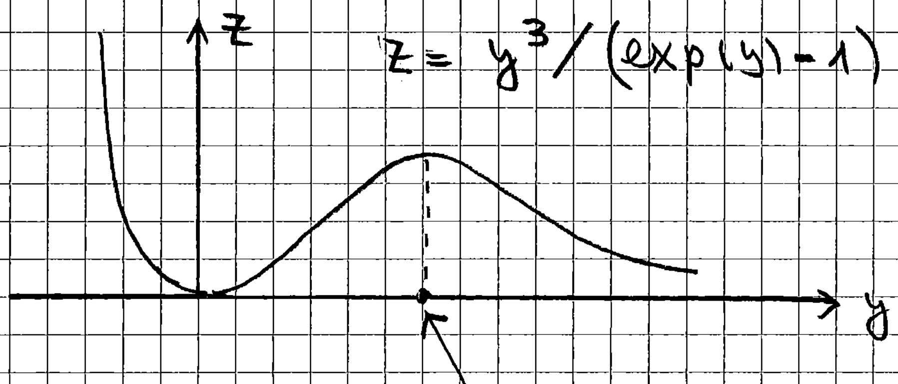
$$+3-\exp(y)\Big] \qquad (5.1.10)$$

Evaluated at an y_* satisfying (5.1.9) we get

$$f''(y_*) = \frac{y_*^2}{(\exp(y_*)-1)^2}\,(3-\exp(y_*))$$

$$= \left[\frac{y_*}{\exp(y_*)-1}\right]^2 \cdot 3 \cdot \left(\frac{2-y_*}{3-y_*}\right) \qquad (5.1.11)$$

For $y_* = 0$ $f''(y_*) = 0$ and since – formally – $f(y) > 0$ for $X > 0$ and $X < 0$ we conclude that $y_* = 0$ is a minimum in which we are not interested (for us $\nu > 0$). Hence $y_* > 0$ must correspond to a maximum.



(5.1.12)

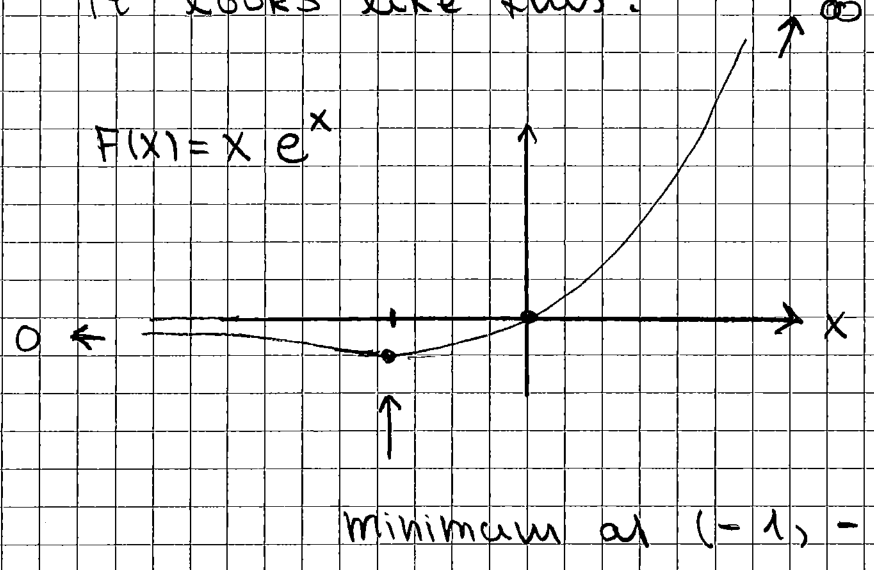
Where is the maximum? It obeys (5.1.9)

$$(y-3)\,\exp(y-3) = -\frac{3}{e^3} \qquad (5.1.13)$$

Consider the function

$$F(X) := X\,\exp(X) \qquad (5.1.14)$$

It looks like this:



(5.1.15)

$$F'(x) = e^x(1+x) = 0 \Leftrightarrow x = -1$$
$$F''(x) = e^x(2+x) \Rightarrow F''(x=-1) > 0$$
(5.1.16)

As F is not monotonous, it has no global inverse; rather it has two branches of inverse functions, called the Lambert W_0 and W_{-1} functions, with

$$W_0 = \left(F\big|_{x \geq -1}\right)^{-1}$$
$$: \left[-\frac{1}{e}, \infty\right) \to [-1, \infty)$$
monotonously increasind

(called the "principal branch")

(5.1.17)

$$W_{-1} = \left(F\big|_{x \leq -1}\right)^{-1} : \left[-\frac{1}{e}, 0\right) \to [-1, -\infty) \quad \text{monotonously decreasing} \tag{5.1.18}$$

The equation

$$F(x) = y \tag{5.1.19}$$

can be solved for given y if $y \geq -1/e$, with two solutions $W_0(y)$ and $W_{-1}(y)$ if $y \in (-\frac{1}{e}, 0)$ and one solution if $y \in [0, \infty)$. For $y = -\frac{1}{e}$ the solutions $W_0(y)$ and $W_{-1}(y)$ coincide.

Now, we want to solve (5.1.9)

$$(y-3) \exp(y-3) = -\frac{3}{e^3} \tag{5.1.9}$$

i.e.

$$F(y-3) = -\frac{3}{e^3} \tag{5.1.20}$$

Since the right-hand side is negative and

$$-\frac{3}{e^3} > -\frac{1}{e} \tag{5.1.21}$$

there are two solutions

$$y_1 = 3 + W_0\left(-\frac{3}{e^3}\right) \qquad (5.1.22a)$$

$$y_2 = 3 + W_{-1}\left(-\frac{3}{e^3}\right) \qquad (5.1.22b)$$

Since one solution of (5.1.9) is $y = 0$ this must correspond to y_2, since W_{-1} takes values in $[-1, -\infty)$ (including -3) whereas W_0 takes values in $[-1, \infty)$ (not including -3).

Hence the solution we want is

$$y = y_* := 3 + W_0(-3/e^3). \qquad (5.1.23)$$

Using "Wolfram Alpha" or the like, get

$$W_0(-3/e^3) = -0.178561 \qquad (5.1.24)$$

$$\Rightarrow \quad y_* \cong 2.82 \quad \text{(rounded)} \qquad (5.1.25)$$

Hence, for Plancks law, the frequency at which $\mathcal{E}_\nu$ has its maximum is

$$y_* = \frac{h\nu_*}{kT} = 2.82 \qquad (5.1.26)$$

$$\text{or} \quad h\nu_* = 2.82 \cdot kT \qquad (5.1.27)$$

If we want this to be 13.6 eV we must have

$$T_1 = \frac{h\nu_*}{2.82\,k} = \frac{13.6\,\text{eV}}{2.82\,k} \tag{5.1.28}$$

k = Boltzmann constant

$$\left.\begin{aligned} &= 1.38 \cdot 10^{-23}\ \text{J} \cdot \text{K}^{-1} \\ &= 8.62 \cdot 10^{-5}\ \text{eV} \cdot \text{K}^{-1} \end{aligned}\right\} \tag{5.1.29}$$

Hence

$$T_1 = \frac{13.6\ \text{eV}}{2.82 \cdot 8.62}\, 10^5\ \text{K}$$

$$= 5.6 \cdot 10^4\ \text{K} \tag{5.1.30}$$

This is the temperature at which black-body-radiation has its maximal density at the ionisation frequency of hydrogen. From (5.1.3) with (5.1.1) we get

$$z = \frac{5.6 \cdot 10^4}{2.73} - 1 = 2.05 \cdot 10^4 \tag{5.1.31}$$

This is 20-times larger than z_{re} (recombination), for we calculated T at which ν <u>max</u> is 13.6 eV.

Problem 2

For $\Omega_{rad} = 1$ and all other Ω's equal to zero we have

$$\left(\frac{dx}{d\lambda}\right)^2 - \frac{1}{x^2} = 0 \qquad (5.2.1)$$

$$\curvearrowright \quad dx\, x = d\lambda$$

$$x(\lambda) = \sqrt{2\lambda}$$

$$\text{or} \quad a(t) = a_0 \left(2 H_0 t\right)^{1/2}$$

$$= A\, t^{1/2}$$

$$\text{with} \quad A = a_0 \sqrt{2 H_0} \,. \qquad \Bigg\} (5.2.2)$$

$$\text{Hence} \quad H(t) = \frac{1}{2} t^{-1} \qquad (5.2.3)$$

$$q(t) = q = -\frac{\ddot{a}\, a}{\dot{a}^2} = 1 \qquad (5.2.4)$$

In lecture 9 we calculated (9.78):

$$\Delta\varphi\,(D = R_{re}, z_{re}) = R_{re} / d_A(z_{re}) \qquad (5.2.5)$$

with

$$d_A(z_{re}) = d_G^{(1)}$$

$$= d_G^{(0)} / (1+z) \Big|_{z = z_{re}}$$

$$= L(t_0, t_{re}) / (1 + z_{re}) \qquad (5.2.6)$$

Hence

$$\Delta\varphi(D = R_{re}, z_{re}) = (1+z_{re}) \frac{R_{re}}{L(t_0, t_{re})} \qquad (5.2.7)$$

We took R_{re} to be the particle horizon at recombination, i.e.

$$R_{re} = L(t_{re}, 0) \qquad (5.2.8)$$

so that

$$\Delta\varphi = (1+z_{re}) \cdot \frac{L(t_{re}, 0)}{L(t_0, t_{re})} \qquad (5.2.9)$$

We evaluated both, $L(t_{re}, 0)$ and $L(t_0, t_{re})$ for the matter dominated model, $a(t) \sim t^{2/3}$.
But generally, for $q(t) = q = \text{const}$ models which are (compare (9.53))

$$a(t) = K \; t^{\frac{1}{1+q}} \qquad (5.2.10)$$

have from (9.56)

$$L(t_0, t_1) = c\, t_0 \frac{1+q}{q}\left[1 - \left(\frac{t_1}{t_0}\right)^{q/1+q}\right] \qquad (5.2.11)$$

For $q = 1/2$ (matter dom.) have

$$L(t_{re}, 0) = 3\, c\, t_{re} \qquad (5.2.12)$$

$$L(t_0, t_{re}) = 3\, c\, t_0 \left[1 - \left(\frac{t_{re}}{t_0}\right)^{1/3}\right] \qquad (5.2.13)$$

Hence

$$\Delta\varphi = \frac{(1+z_{re})}{\left[1-\left(\frac{t_{re}}{t_0}\right)^{1/3}\right]}\left(\frac{t_{re}}{t_0}\right) \qquad (5.2.14)$$

And with

$$(1+z) = \frac{a(t_0)}{a(t_1)} = \left(\frac{t_0}{t_1}\right)^{\frac{1}{1+q}}$$

$$\Rightarrow \qquad \frac{t_0}{t_{re}} = (1+z)^{(1+q)} \qquad (5.2.15)$$

$$= (1+z)^{3/2} \quad \text{for } q = \frac{1}{2} \qquad (5.2.16)$$

we arrived at

$$\Delta\varphi = \frac{(1+z_{re})\cdot(1+z_{re})^{-3/2}}{1-(1+z_{re})^{-1/2}}$$

$$= \frac{1}{\sqrt{1+z_{re}}-1} \qquad (5.2.17)$$

Now, if we replace the calculation of $L(t_{re}, 0)$ with that of the radiation dominated model, we get from (5.2.11) instead of (5.2.12) for $q=1$:

$$L^{RD}(t_{re}, 0) = 2 c t_0 \tag{5.2.18}$$

But we maintain the calculation for $L(t_0, t_{re})$ in the matter dominated model $(q = \frac{1}{2})$, since matter dominance sets in long before t_{re} (compare Sheet 4, Problem 4). Then

$$\Delta\varphi = (1+z_{re}) \frac{L^{RD}(t_{re}, 0)}{L^{MD}(t_0, t_{re})} \tag{5.2.19}$$

(RD = Radiation Dominated
MD = Matter Dominated)

$$\Delta\varphi = (1+z_{re}) = \frac{2}{3} \frac{1}{\sqrt{1+z_{re}} - 1} \tag{5.2.20}$$

So the only difference of replacing the dynamics between Big-Bang and recombination by radiation dominance is that $\Delta\varphi$ gets multiplied by $2/3$, i.e. becomes yet smaller.

Problem 3

We recall from Lecture 6 the dimensionless Friedmann-Equation (6.53a-c)

$$\left(\frac{dX}{d\lambda}\right)^2 + V(X) = E \qquad (5.3.1)$$

where $\lambda = t H_0$

$$X(\lambda) = a(\lambda H_0)/a_0 \qquad (5.3.2)$$

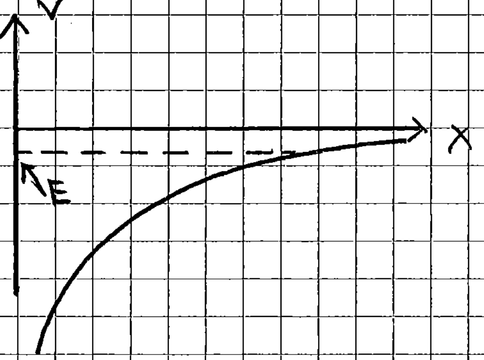
$$V(X) = -\frac{\Omega_{rad}}{X^2} - \frac{\Omega_{dust}}{X} - \Omega_\Lambda X^2 \qquad (5.3.3)$$

$$E = \Omega_k \qquad (5.3.4)$$

and $\Omega_{rad} + \Omega_{dust} + \Omega_\Lambda + \Omega_k = 1 \qquad (5.3.5)$

For $\Omega_\Lambda = 0$ $V(X)$ is always negative and approaches $V=0$ for $X \to \infty$.

For $k=+1$,
i.e. $\Omega_k < 0$
we have $E<0$.
Hence "particle" rolls up and down again the potential hill.



Note that

$$\Omega_k := -\frac{kc^2}{H_0^2 a_0^2} < 0 \qquad (5.3.6)$$

so that for $k = +1$ have

$$\Omega_k = -\frac{c^2}{H_0^2 a_0^2} \tag{5.3.7}$$

and for $\Omega_\Lambda = \Omega_{rad} = 0$

$$\Omega_{dust} = 1 - \Omega_k > 1$$

$$\parallel$$

$$\frac{8\pi G \rho_0}{3 H_0^2} \Rightarrow \rho_0 > \frac{3 H_0^2}{8\pi G} =: \rho_{crit} \tag{5.3.8}$$

In case $\Omega_\Lambda = \Omega_{rad} = 0$ we write

$$\Omega := \Omega_{dust} \tag{5.3.9}$$

$$\Omega_k = 1 - \Omega \quad \text{(cosm. triangle)} \tag{5.3.10}$$

and (5.3.1) becomes

$$\left(\frac{dx}{d\lambda}\right)^2 = \frac{\Omega}{x} + (1 - \Omega) \tag{5.3.11}$$

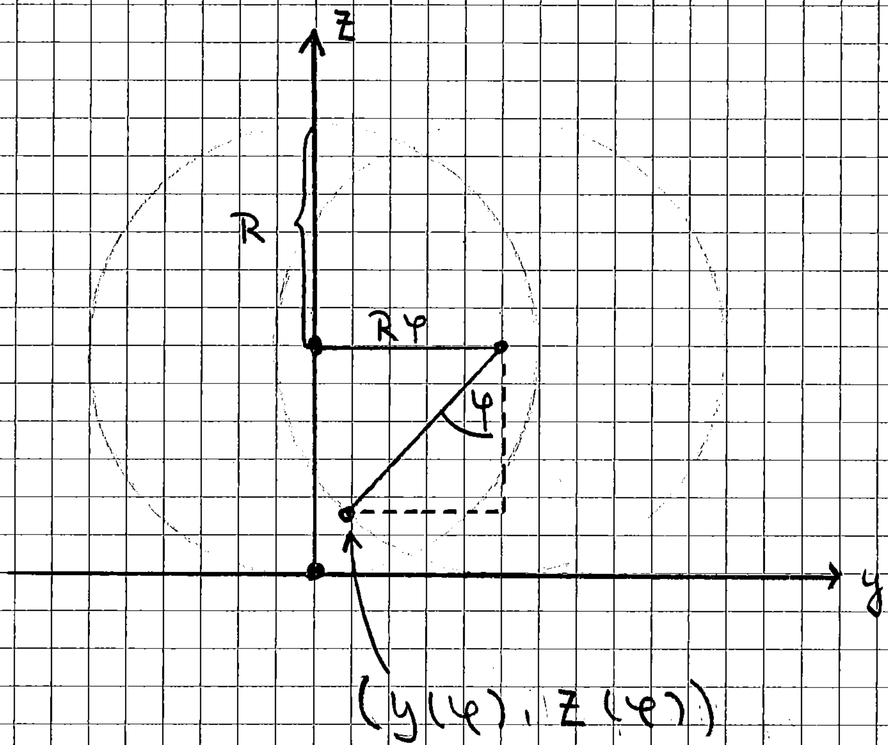
$$\Leftrightarrow \quad \frac{1}{\Omega - 1}\left(\frac{dx}{d\lambda}\right)^2 = \frac{\Omega}{\Omega - 1}\frac{1}{x} - 1$$

$$\Leftrightarrow \quad \left(\frac{dx}{d\sigma}\right)^2 = \frac{2R}{x} - 1 \tag{5.3.12}$$

where $$\sigma := \sqrt{\Omega - 1}\,\lambda \tag{5.3.13}$$

$$R := \frac{1}{2}\frac{\Omega}{\Omega - 1} \tag{5.3.14}$$

We recall once more the "cycloid", i.e. the curve described by a fixed point on the circumference of a rolling wheel



(5.3.15)

$$\left.\begin{aligned} y(\varphi) &= R(\varphi - \sin\varphi) \\ z(\varphi) &= R(1-\cos\varphi) \end{aligned}\right\} \quad (5.3.16)$$

$$\frac{dz}{dy} = \frac{dz/d\varphi}{dy/d\varphi} = \frac{\sin\varphi}{1-\cos\varphi}$$

$$= \frac{(1-\cos^2\varphi)^{1/2}}{1-\cos\varphi} = \left[\frac{1+\cos\varphi}{1-\cos\varphi}\right]^{1/2}$$

$$= \left[\frac{-1+\cos\varphi+2}{1-\cos\varphi}\right]^{1/2} = \left(\frac{2R}{z}-1\right)^{1/2} \quad (5.3.17)$$

The differential equation for the graph $z = z(y)$ of the cycloid is

$$\frac{dz}{dy} = \left(\frac{2R}{z} - 1\right)^{1/2} \tag{5.3.18}$$

In our case

$$\frac{dx}{d\sigma} = \left(\frac{2R}{x} - 1\right)^{1/2} \tag{5.3.19}$$

so that we read off parametric solution

$$\left.\begin{aligned} \sigma &= \sigma(\varphi) = R(\varphi - \sin\varphi) \\ x &= x(\varphi) = R(1 - \cos\varphi) \end{aligned}\right\} \tag{5.3.20}$$

$$\left.\begin{aligned} \text{where} \quad x = \frac{a}{a_0}, \quad \sigma &= \sqrt{\Omega - 1}\ \lambda \\ R &= \frac{1}{2}\,\frac{\Omega}{\Omega - 1} \end{aligned}\right\} \tag{5.3.21}$$

In terms of t and $a(t)$ this reads:

$$t(\varphi) = \frac{1}{2H_0}\,\frac{\Omega}{(\Omega - 1)^{3/2}}\,(\varphi - \sin\varphi) \tag{5.3.22a}$$

$$a(\varphi) = \frac{a_0}{2}\,\frac{\Omega}{\Omega - 1}\,(1 - \cos\varphi) \tag{5.3.22b}$$

So the universe starts at $\varphi = 0$, $t = 0$ at a "Big-Bang", expand to a maximal size at $\varphi = \pi$, i.e.

$$t_{max} = \frac{\pi}{2H_0} \frac{\Omega}{(\Omega - 1)^{3/2}} \quad (5.3.23a)$$

$$a_{max} = a_0 \frac{\Omega}{\Omega - 1} \quad (5.3.23b)$$

and then recollapse at a "Big-Crunch" at $\varphi = 2\pi$

$$t_* = 2\, t_{max} = \frac{\pi}{H_0} \frac{\Omega}{(\Omega - 1)^{3/2}} \quad (5.3.24)$$

where the time from Big-Bang to Big-Crunch is the "lifetime" t_*.

The distance a light-signal travels is given by

$$d\chi = c \frac{dt}{a(t)} = c \frac{da}{\dot{a}\, a}$$

$$= c \frac{da}{H a^2} = \frac{dx}{H(x) x^2} \frac{c}{a_0} \quad (5.3.25)$$

$$H(x) = H_0 \left(\Omega_{rad} x^{-4} + \Omega_{dust} x^{-3} + \Omega_k x^{-2} + \Omega_\Lambda \right)^{1/2}$$

$$= \frac{H_0}{x^2} \left(\Omega x - (\Omega - 1) x^2 \right)^{1/2}$$

Hence

$$X^2 H(X) = H_0 \left(\Omega X - (\Omega-1) X^2 \right)^{1/2}$$

$$= H_0 (\Omega-1)^{1/2} \left[-X^2 + \frac{\Omega}{\Omega-1} X \right]^{1/2}$$

$$= H_0 (\Omega-1)^{1/2} \left[-\left(X - \frac{\Omega}{2(\Omega-1)} \right)^2 + \left(\frac{\Omega}{2(\Omega-1)} \right)^2 \right]^{1/2}$$

$$= H_0 \frac{\Omega}{2} \frac{1}{\sqrt{\Omega-1}} \left[-\left(\frac{2(\Omega-1)}{\Omega} X - 1 \right)^2 + 1 \right]^{1/2} \quad (5.3.26)$$

Inserting this into (5.3.25) we get

$$d\chi = \frac{2\sqrt{\Omega-1}}{H_0 \Omega a_0} c \cdot \frac{dX}{\left[-\left(\frac{2(\Omega-1)}{\Omega} X - 1 \right)^2 + 1 \right]^{1/2}}$$

$$= \frac{c}{H_0 a_0 \sqrt{\Omega-1}} \frac{dz}{\sqrt{1-z^2}} \quad (5.3.27)$$

with

$$z := \left(\frac{2(\Omega-1)}{\Omega} X - 1 \right) \quad (5.3.28)$$

Interating from $\chi = 0$ for $X = 0$ ($a = 0$, Big-Bang) or $z = -1$ to χ for X, i.e. $z = \frac{2(\Omega-1)}{\Omega} X - 1$, we get, using

$$\int \frac{dz}{\sqrt{1-z^2}} = \arcsin(z) \quad (5.3.29)$$

$$\chi = \frac{c}{H_0 a_0 \sqrt{\Omega - 1}} \arcsin(z) \Big|_{-1}^{\frac{2(\Omega-1)}{\Omega} x - 1} \qquad (5.3.30)$$

Using (5.3.10) and (5.3.7), i.e.

$$\Omega - 1 = -\Omega_k = \frac{c^2}{H_0^2 a_0^2}$$

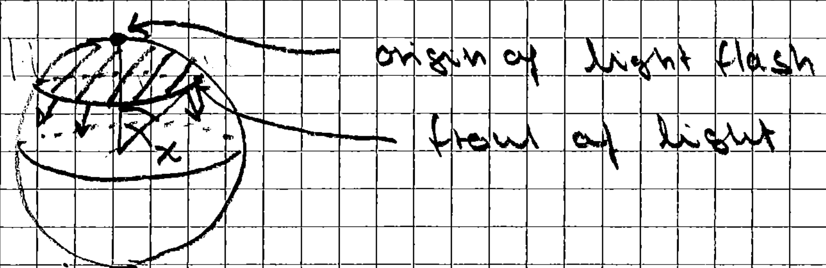
so that

$$\frac{c}{H_0 a_0 \sqrt{\Omega - 1}} = 1 \qquad (5.3.31)$$

we get

$$\boxed{\chi = \frac{\pi}{2} + \arcsin\left(\frac{2(\Omega-1)}{\Omega} x - 1\right)} \qquad (5.3.32)$$

This formula gives the polar angle χ of the spherical cap on S^3 covered by the light-flash that started at $\chi = 0$ at $x = 0$ ($a = 0$).



The semi-sphere ("half the universe") is covered for $\chi = \pi/2$

This happens for

$$X = \frac{\Omega}{2(\Omega-1)} \tag{5.3.33}$$

$$\text{i.e.} \quad a = \frac{a_0}{2} \frac{\Omega}{\Omega-1} \tag{5.3.34}$$

which by (5.3.22b) means $\varphi = \frac{\pi}{2}$
and hence

$$t\left(\varphi = \frac{\pi}{4}\right) = \frac{1}{2H_0} \frac{\Omega}{(\Omega-1)^{3/2}} \left(\frac{\pi}{2} - 1\right) \tag{5.3.35}$$

The 3-sphere is fully covered by the light flash for $\chi = \pi$, i.e.

$$\frac{2(\Omega-1)}{\Omega} x - 1 = 1$$

$$\rightarrow \quad X = \frac{\Omega}{\Omega-1}$$

$$\text{or} \quad a = a_0 \ \Omega/(\Omega-1) \tag{5.3.36}$$

which is for $\varphi = \pi$ and

$$t = t_{max} = \frac{\pi}{2H_0} \ \frac{\Omega}{(\Omega-1)^{3/2}} \tag{5.3.37}$$

That is, in a dust-dominated universe can the observer see the whole universe at the time of maximal expansion.

Now we repeat the same analysis for a radiation - driven dynamics. This time we set

$$\Omega_{dust} = \Omega_\Lambda = 0$$

$$\Omega_{rad} =: \Omega \quad > 1$$

$$\Omega_k = 1 - \Omega \quad < 0 \qquad (5.3.38)$$

The Friedmann - Equation (5.3.1) then is

$$\left(\frac{dx}{d\lambda}\right)^2 = \frac{\Omega}{x^2} - (\Omega - 1)$$

$$\frac{dx}{\left[\frac{\Omega}{x^2} - (\Omega-1)\right]^{1/2}} = d\lambda$$

$$\frac{x\,dx}{\left[-(\Omega-1)x^2 + \Omega\right]^{1/2}} = d\lambda \qquad (5.3.39)$$

Straight forward integration with $x = 0$ for $\lambda = 0$ gives

$$-\frac{1}{(\Omega-1)}\left[-(\Omega-1)x^2 + \Omega\right]^{1/2} + \frac{\Omega^{1/2}}{\Omega - 1} = \lambda$$

$$\Rightarrow -\left[-(\Omega-1)x^2+\Omega\right]^{1/2} = \left((\Omega-1)\lambda - \Omega^{1/2}\right)$$

$$-(\Omega-1)x^2+\Omega = \left((\Omega-1)\lambda-\Omega^{1/2}\right)^2$$

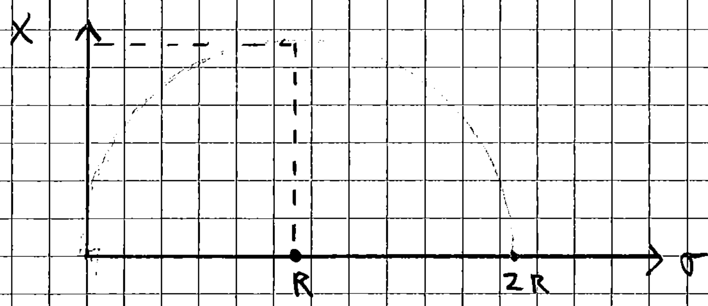
$$\Leftrightarrow \quad x^2+\left[\lambda(1-\Omega)^{1/2}-\left(\frac{\Omega}{\Omega-1}\right)^{1/2}\right]^2=\frac{\Omega}{\Omega-1}$$

$$\Leftrightarrow \quad x^2+\left[\sigma-\left(\frac{\Omega}{\Omega-1}\right)^{1/2}\right]^2=\frac{\Omega}{\Omega-1} \qquad (5.3.40)$$

This describes a circle of radius

$$R=\left(\frac{\Omega}{\Omega-1}\right)^{1/2} \qquad (5.3.41)$$

and centre $(R,0)$ in the σ-x plane of which only the upper semicircle $x \geq 0$ is relevant here



(5.4.42)

We set

$$\sigma=(\Omega-1)^{1/2}\lambda \qquad (5.4.43)$$

Maximal expansion is at $\sigma=R$ with $x=R$, i.e.

lifetime

$$t_{max}=\frac{1}{H_0}\,\frac{\Omega^{1/2}}{\Omega-1}=\frac{1}{2}t_* \qquad (5.3.44a)$$

$$a_{max}=a_0\left(\frac{\Omega}{\Omega-1}\right)^{1/2} \qquad (5.3.44b)$$

Regarding the light flash we get, instead of (5.3.26)

$$x^2 H(x) = H_0 \left(\Omega - (\Omega - 1) x^2 \right)^{1/2} \qquad (5.3.45)$$

hence

$$d\chi = \frac{c}{a_0 H_0} \frac{dx}{\left[\Omega - (\Omega - 1) x^2 \right]^{1/2}}$$

$$= \frac{c}{a_0 H_0} \left(\frac{1}{\Omega} \right)^{1/2} \frac{dx}{\left[1 - \frac{\Omega - 1}{\Omega} x^2 \right]^{1/2}}$$

$$= \frac{c}{a_0 H_0} \left(\frac{1}{\Omega - 1} \right)^{1/2} \frac{dz}{\sqrt{1 - z^2}} \qquad (5.3.46)$$

where
$$z = \left(\frac{\Omega - 1}{\Omega} \right)^{1/2} x \qquad (5.3.47)$$

Again (5.3.31) holds (since $\Omega - 1 = -\Omega k = c^2 / H_0^2 a_0^2$), so that

$$\chi = \arcsin(z) \Big|_0^{\left(\frac{\Omega - 1}{\Omega} \right)^{1/2} x}$$

$$= \arcsin \left[\left(\frac{\Omega - 1}{\Omega} \right)^{1/2} x \right] \qquad (5.3.48)$$

This is the semi-sphere $\chi = \frac{\pi}{2}$ for

$$x = \left(\frac{\Omega}{\Omega - 1} \right)^{1/2} \qquad (5.3.49)$$

i.e.

$$a = a_{max} = a_0 \left(\frac{\Omega}{\Omega - 1} \right)^{1/2} \tag{5.3.50}$$

and twice as much, i.e. $\chi = \pi$, corresponding to the whole S^3, if the z-integration extends from $z = 0$ (Big-Bang) to $z = 1$ (maximal extent) and back to $z = 0$ (Big-Crunch).

So for the radiation-dominated universe the observer sees only half of the universe at $a = a_{max}$ and has to wait until the Big-Crunch in order to see all of it!