Sheet 6: Solutions

Problem 1

$$\dot{a}^2(t) = \frac{8\pi G}{3} \rho(t) a^2(t) + \frac{\Lambda c^2}{3} a^2(t) - kc^2 \qquad (6.1.1)$$

We define:

$$\Omega_m(t) := \frac{8\pi G \rho(t)}{3H^2(t)}, \qquad (6.1.2a)$$

$$\Omega_\Lambda(t) := \frac{\Lambda c^2}{3H^2(t)}, \qquad (6.1.2b)$$

$$\Omega_k := \frac{-kc^2}{H^2(t) a^2(t)}. \qquad (6.1.2c)$$

Where

$$\rho = \rho_{rad} + \rho_{dust} \qquad (6.1.3)$$

and

$$\rho_{rad}(t) a^4(t) = \text{const} \qquad (6.1.4a)$$

$$\rho_{dust}(t) a^3(t) = \text{const}. \qquad (6.1.4b)$$

Now, equation (6.1.1) is equivalent to

$$H^2(t) = \frac{8\pi G}{3} \underbrace{\rho(t)}_{\rho_{rad} + \rho_{dust}} + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2(t)}, \qquad (6.1.5)$$

which just says that

$$1 = \Omega_{rad}(t) + \Omega_{dust}(t) + \Omega_\Lambda(t) + \Omega_K(t); \qquad (6.1.6)$$

Another way to write (6.1.5) is as follows: Let a sub- or superscript "0" denote evaluation at $t = t_0$; then dropping the argument t for the t-dependent quantities, we rewrite (6.1.5) as

$$\left(\frac{H}{H_0}\right)^2 = \Omega^{(0)}_{rad} \frac{\rho_{rad}}{\rho^{(0)}_{rad}} + \Omega^{(0)}_{dust} \frac{\rho_{dust}}{\rho^{(0)}_{dust}} + \Omega^{(0)}_\Lambda + \Omega^{(0)}_K \left(\frac{a_0}{a}\right)^2 \qquad (6.1.7)$$

From Lecture 6, formulae (6.23-24) we know that

$$\rho_{rad}\, a^4 = \rho^{(0)}_{rad}\, a_0^4 \qquad (6.1.8a)$$

$$\rho_{dust}\, a^3 = \rho^{(0)}_{dust}\, a_0^3 \qquad (6.1.8b)$$

Inserting this into (6.1.7) gives

$$\left(\frac{H}{H_0}\right)^2 = \Omega^{(0)}_{\text{rad}} \left(\frac{a_0}{a}\right)^4 + \Omega^{(0)}_{\text{dust}} \left(\frac{a_0}{a}\right)^3 + \Omega^{(0)}_{k} \left(\frac{a_0}{a}\right)^2 + \Omega^{(0)}_{\Lambda} \qquad (6.1.9)$$

Now, from the definitions (6.1.2) we read off, using (6.1.8),

$$\Omega_{\text{rad}} = \Omega^{(0)}_{\text{rad}} \left(\frac{H_0}{H}\right)^2 \left(\frac{a_0}{a}\right)^4 \qquad (6.1.10a)$$

$$\Omega_{\text{dust}} = \Omega^{(0)}_{\text{dust}} \left(\frac{H_0}{H}\right)^2 \left(\frac{a_0}{a}\right)^3 \qquad (6.1.10b)$$

$$\Omega_{k} = \Omega^{(0)}_{k} \left(\frac{H_0}{H}\right)^2 \left(\frac{a_0}{a}\right)^2 \qquad (6.1.10c)$$

$$\Omega_{\Lambda} = \Omega^{(0)}_{\Lambda} \left(\frac{H_0}{H}\right)^2 \qquad (6.1.10d)$$

Hence, with (6.1.9),

$$\Omega_{\text{rad}} = \frac{\Omega^{(0)}_{\text{rad}}}{\Omega^{(0)}_{\text{rad}} + \Omega^{(0)}_{\text{dust}} \left(\frac{a}{a_0}\right) + \Omega^{(0)}_{k} \left(\frac{a}{a_0}\right)^2 + \Omega_{\Lambda} \left(\frac{a}{a_0}\right)^3} \qquad (6.1.11a)$$

$$\Omega_{\text{dust}} = \frac{\Omega^{(0)}_{\text{dust}}}{\Omega^{(0)}_{\text{rad}} \left(\frac{a_0}{a}\right) + \Omega^{(0)}_{\text{dust}} + \Omega^{(0)}_{k} \left(\frac{a}{a_0}\right) + \Omega^{(0)}_{\Lambda} \left(\frac{a}{a_0}\right)^2} \qquad (6.1.11b)$$

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$$\Omega_k = \frac{\Omega_k^{(0)}}{\Omega_{rad}^{(0)}\left(\frac{a_0}{a}\right)^2 + \Omega_{dust}^{(0)}\left(\frac{a_0}{a}\right) + \Omega_k^{(0)} + \Omega_\Lambda^{(0)}\left(\frac{a}{a_0}\right)} \qquad (6.1.11c)$$

$$\Omega_\Lambda = \frac{\Omega_\Lambda^{(0)}}{\Omega_{rad}^{(0)}\left(\frac{a_0}{a}\right)^4 + \Omega_{dust}^{(0)}\left(\frac{a_0}{a}\right)^3 + \Omega_k^{(0)}\left(\frac{a_0}{a}\right)^2 + \Omega_\Lambda^{(0)}} \qquad (6.1.11d)$$

Problem 2

We first recall the heuristics behind m_P and l_P

m_P : Compton wavelength of mass m

$$\lambda_c = \frac{h}{mc} \tag{6.2.1}$$

Schwarzschild-Radius of mass m

$$R_s = \frac{2Gm}{c^2} \tag{6.2.2}$$

$$\lambda_c > R_s \Leftrightarrow \frac{h}{mc} > \frac{2Gm}{c^2}$$

$$\Rightarrow m < \sqrt{\frac{hc}{2G}} = \sqrt{\pi}\; m_P \tag{6.2.3}$$

or

$$\frac{\lambda_c}{\pi} > R_s \Leftrightarrow m < m_P \tag{6.2.4}$$

$\Rightarrow$ "Black-Holes" of mass $m < m_P$ are genuine "Quantum Objects" [Presumably it will make no sense to claim that a mass $m < m_P$ is a Black-Hole.]

l_p: In order to measure a distance l you have to shine light on it with wavelength $\lambda/2 < l$. A single photon of that wavelength has energy

$$E = h\nu = h\frac{c}{\lambda} > \frac{hc}{2l} \qquad (6.2.5)$$

corresponding to a mass

$$m = \frac{E}{c^2} > \frac{h}{2lc} \qquad (6.2.6)$$

and a Schwarzschild - Radius

$$R_s = \frac{2Gm}{c^2} = \frac{Gh}{lc^3} \qquad (6.2.7)$$

If $R_s > l$ then we create a Black-Hole through the measurement. This happens if

$$\frac{Gh}{lc^3} > l \iff l < \sqrt{\frac{Gh}{c^3}} \qquad (6.2.8)$$

or $\quad l < \sqrt{2\pi}\, l_p \qquad (6.2.9)$

Presumably this means that lenghts smaller than l_p make no (operational) sense.

Currently

$$\Omega^{(0)}_{dust} \cong 0.3 \ , \ \Omega^{(0)}_{rad} \cong 10^{-4} \qquad (6.2.10)$$

$$\Omega^{(0)}_{\Lambda} \cong 0.7 \ , \ \Omega^{(0)}_{k} \cong 0 \qquad (6.2.11)$$

From (6.1.11) we infer that for $a \ll a_0$ the dominant contributions are

$$\Omega_{rad} = 1 - \frac{\Omega^{(0)}_{dust}}{\Omega^{(0)}_{rad}} \left(\frac{a}{a_0}\right) + \cdots \qquad (6.2.12a)$$

$$\Omega_{dust} = \frac{\Omega^{(0)}_{dust}}{\Omega^{(0)}_{rad}} \left(\frac{a}{a_0}\right) + \cdots \qquad (6.2.12b)$$

$$\Omega_{k} = \frac{\Omega^{(0)}_{k}}{\Omega^{(0)}_{rad}} \left(\frac{a}{a_0}\right)^2 \qquad (6.2.13c)$$

$$\Omega_{\Lambda} = \frac{\Omega^{(0)}_{\Lambda}}{\Omega^{(0)}_{rad}} \left(\frac{a}{a_0}\right)^4 + \cdots \qquad (6.2.13d)$$

where $\cdots$ means terms of higher power in (a/a_0) than those already written.

Since $\frac{\Omega^{(0)}_{dust}}{\Omega^{(0)}_{rad}} \simeq 10^{4}$ and $\qquad (6.2.14)$

$(a/a_0) \ll 10^{-4}$ we are deeply in the radiation dominated phase an can safely forget all Ω's but Ω_{rad}.

Radiation dominance means that

$$\left(\frac{dx}{d\lambda}\right)^2 + V(x) = E$$

$$V(x) = -\frac{\Omega_{rad}^{(0)}}{x^2} - \frac{\Omega_{dust}^{(0)}}{x} - \Omega_\Lambda x^2$$

$$E = \Omega_\kappa$$

with all $\Omega = 0$ except

$$\Omega_{rad} = 1$$

Hence

$$\left(\frac{dx}{d\lambda}\right)^2 = \frac{\Omega_{rad}^{(0)}}{x^2} \qquad (6.2.15)$$

$$\Rightarrow \quad x\,dx = \sqrt{\Omega_{rad}^{(0)}}\, d\lambda$$

$$\int_0^{\lambda} x'\,dx' = \int_{\lambda=0}^{\lambda} \sqrt{\Omega_{rad}^{(0)}}\, d\lambda'$$

$$\Rightarrow \quad x^2(\lambda) = 2\cdot\sqrt{\Omega_{rad}^{(0)}}\,\lambda$$

where $x = a/a_0$, $\lambda = H_0 t$.

Hence

$$\left(\frac{a(t)}{a_0}\right)^2 = 2\cdot\sqrt{\Omega_{rad}^{(0)}}\,(H_0 t) \qquad (6.2.16)$$

for all times $t > 0$ for which radiation dominance holds.

In particular it holds for $t = t_P$:

$$\left(\frac{a_p}{a_0}\right)^2 = 2 \cdot \sqrt{\Omega^{(0)}_{rad}}\,(H_0\, t_p)$$

$$= 2 \cdot \sqrt{\Omega^{(0)}_{rad}}\,(l_p / R_H) \qquad (6.2.17)$$

Where $R_H := \frac{c}{H_0}$ (6.2.18)

$$l_p = c \cdot t_p \qquad (6.2.19)$$

Using (6.1.11c) we get for Ω_k at time t_p

$$\frac{\Omega_k(t_p)}{\Omega_k(t_0)} = \frac{1}{\Omega^{(0)}_{rad}\left(\frac{a_0}{a_p}\right)^2}$$

$$= \frac{1}{\Omega^{(0)}_{rad}}\left(\frac{a_p}{a_0}\right)^2$$

$$= \frac{2}{\sqrt{\Omega^{(0)}_{rad}}}\left(\frac{l_p}{R_H}\right) \qquad (6.2.20)$$

Have

$$\left.\begin{aligned} l_p &= 1.616 \times 10^{-35}\,\mathrm{m} \\ R_H &= 1.3 \times 10^{26}\,\mathrm{m} \\ \Omega^{(0)}_{rad} &= 8.25 \times 10^{-5} \end{aligned}\right\} \qquad (6.2.21)$$

$$\Rightarrow \frac{\Omega_k(t_p)}{\Omega_k(t_0)} = 2.74 \times 10^{-59} \qquad (6.2.22)$$

Problem 3

We assume that for $t < t_P$ a time period to exist in which the expansion is exponential:

$$a(t) = a_0 \exp(H(t-t_0)) \qquad (6.3.1)$$

where

$$H = \frac{\dot{a}(t)}{a(t)} = \text{const.} \qquad (6.3.2)$$

and t_0 is some "initial time" $< t_P$. From (6.1.2c)

$$\Omega_K(t) = -\frac{k c^2}{H^2(t)\, a^2(t)}$$

$$= -\frac{k c^2}{H^2 a_0^2} \exp\left(-2H(t-t_0)\right) \qquad (6.3.3)$$

$$\Rightarrow \frac{\Omega_K(t)}{\Omega_K(t_0)} = \exp(-2H(t-t_0)). \qquad (6.3.4)$$

So if initially, for $t_0 < t_P$, $\Omega_K(t_0)$ had been of generic order unity, then, in order for $\Omega_K(t_P)$ to be 10^{-59}, we must have

$$\exp(-2H(t-t_0)) = 10^{-59} \qquad (6.3.5)$$

$$\text{or } (t-t_0) = \frac{59}{2} \ln(10) \frac{1}{H} \simeq \frac{68}{H} \qquad (6.3.6)$$