Exercises for the lecture on

Theory of Fundamental Interactions (summer 2022)

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Sheet 1

Problem 1

Consider the Lie-algebra $L = (\mathbb{R}^3, \times)$ and its adjoint representation:

$$L \to \mathrm{End}(\mathbb{R}^3)\,, \quad \vec{x} \mapsto \mathrm{ad}_{\vec{x}} : \vec{y} \mapsto \mathrm{ad}_{\vec{x}}(\vec{y}) = \vec{x} \times \vec{y}\,. \tag{1}$$

Let exp denote the exponential map which is defined by its power-series expansion in $\mathrm{End}(\mathbb{R}^3)$. Show that its image lies in $\mathrm{GL}(\mathbb{R}^3) \subset \mathrm{End}(\mathbb{R}^3)$. (Hint: You have to show that $\det(\exp(T)) \neq 0$ for all $T \in \mathrm{End}(\mathbb{R}^3)$.)

Show by evaluating the exponential series that for $\vec{n} \in \mathbb{R}^3$ with $\|\vec{n}\| = 1$ and $\theta \in \mathbb{R}$

$$\exp\big(\theta\, \mathrm{ad}_{\vec{n}}\big) = P_{\|} + \big(\cos(\theta)\, \mathrm{id}_{\mathbb{R}^3} + \sin(\theta)\, \mathrm{ad}_{\vec{n}}\big) \circ P_{\perp}\,, \tag{2}$$

where $P_{\|} : \vec{x} \mapsto \vec{n}(\vec{n} \cdot \vec{x})$ and $P_{\perp} : \vec{x} \mapsto \vec{x} - \vec{n}(\vec{n} \cdot \vec{x})$ are the projection maps parallel and perpendicular to $\vec{n}$. (Hint: Show first that $\mathrm{ad}_{\vec{n}} \circ \mathrm{ad}_{\vec{n}} = -P_{\perp}$ and decompose the exponential series in even and odd powers.)

Argue that $\exp\big(\theta\, \mathrm{ad}_{\vec{n}}\big)$ is a rotation in $\mathbb{R}^3$ (with standard euclidean inner product) by an angle θ and oriented axis $\vec{n}$.

Problem 2

Let $L := (V, [\cdot, \cdot])$ be a Lie-algebra and $\mathrm{ad} : L \to \mathrm{End}(V)$ given by $X \mapsto \mathrm{ad}_X$ with $\mathrm{ad}_X(Y) := [X, Y]$. Show that this is a homomorphism of Lie-algebras:

$$\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]. \tag{3}$$

Note that in the lecture we had already shown that ad_X is a derivation, i.e. $\mathrm{ad}_X([Y, Z]) = [\mathrm{ad}_X(Y), Z] + [Y, \mathrm{ad}_X(Z)]$.

Problem 3

Let $L := (V, [\cdot, \cdot])$ be a Lie-algebra and $K : V \times V \to \mathbb{K}$ its Killing-form $K(X, Y) := \mathrm{Trace}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$. Show that

$$K([X, Y], Z) + K(Y, [X, Z]) = 0\,. \tag{4}$$

Hint: Use (3) and the cyclic property of the trace.

Problem 4

Consider again the Lie-algebra $L = (\mathbb{R}^3, \times)$. Let $\{e_a \mid a = 1, 2, 3\}$ be an orthonormal basis with respect to the standard euclidean inner product. Compute the components $K_{ab} = K(e_a, e_b)$ of the Killing-form for that basis.

Problem 5

Let $L' = (V', [\cdot,\cdot]')$ and $L'' = (V'', [\cdot,\cdot]'')$ be two Lie-algebras and $\sigma : L'' \to \mathrm{Der}(L')$ a Lie-homomorphism. In the lecture we defined the *semi-direct sum of* L' *with* L'' by $L = (V, [\cdot,\cdot])$ with $V := V' \oplus V''$ and $[X' \oplus X'', Y' \oplus Y''] := ([X', Y']' + \sigma_{X''}(Y') - \sigma_{Y''}(X')) \oplus [X'', Y'']''$. Verify that the Lie-product so defined on V satisfies the Jacobi-identity.

Problem 6

Show that the set of real traceless (2×2)–matrices form a real Lie-algebra of the Lie-product is defined by the commutator. We call ist $\mathfrak{sl}(2, \mathbb{R})$.

Show that

$$X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}$$

is a basis of $\mathfrak{sl}(2, \mathbb{R})$ with

$$[X^+, X^-] = H, \quad [H, X^\pm] = \pm 2\, X^\pm . \tag{6}$$

Show that $\mathfrak{sl}(2, \mathbb{R})$ is simple.

Hint: Let $I \subseteq \mathfrak{sl}(2, \mathbb{R})$ be an ideal and $C = x_+ X^+ + x_- X^- + hH \in I$. Use (6) and proceed as follows: If, e.g., $x_- \neq 0$ then it follows from considering $[X^+, [X^+, C]]$ that $X^+ \in I$, which implies $I = \mathfrak{sl}(2, \mathbb{R})$ (why?). Now, the same conclusion can be drawn if $x_+ \neq 0$ or if $h \neq 0$.

Problem 7

Sei $L = (V, [\cdot,\cdot])$ ve a real Lie-algebra of dimension $\dim(L) := \dim(V) = n$. We consider the complex numbers $\mathbb{C}$ as real vector space of dimension 2. An obvious basis of $\mathbb{C}$ over $\mathbb{R}$ is $\{1, i\}$. We define a new *real* Lie-algebra $L' = (V', [\cdot,\cdot]')$ of dimension $2n$ through

$$V' := \mathbb{C} \otimes_{\mathbb{R}} V \tag{7a}$$

mit Lie-Produkt

$$[z_1 \otimes X_1 \, , \, z_2 \otimes X_2]' := z_1 z_2 \otimes [X_1, X_2] \tag{7b}$$

on elements of the form $z \otimes X$ and bilinear extension on all of L'. (Convince yourself that this makes L' indeed into a Lie-algebra.)

If $\{e_a \mid a = 1, \cdots, n\}$ is a Basis of V then a basis $\{e'_A \mid A = 1, \cdots, 2n\}$ of V' is given by

$$e'_a := 1 \otimes e_a\,, \quad e'_{n+a} := i \otimes e_a\,, \quad (a = 1, \cdots, n)\,. \tag{8}$$

Show that $(a, b, c = 1, \cdots, n)$:

$$[e'_a, e'_b]' \;=\; C^c_{ab}\, e'_c\,, \tag{9a}$$

$$[e'_a, e'_{n+b}]' \;=\; C^c_{ab}\, e'_{n+c}\,, \tag{9b}$$

$$[e'_{n+a}, e'_{n+b}]' \;=\; -C^c_{ab}\, e'_c\,. \tag{9c}$$

Show further that if $K_{ab} = C^m_{an} C^n_{bm}$ are the $(n \times n)$ components of the Killing-form K of L, then the $(2n \times 2n)$ components of the Killing-form K' of L' are given by

$$K'(e'_a\,,\, e'_b) \;=\; 2K_{ab}\,, \tag{10a}$$

$$K'(e'_a\,,\, e'_{n+b}) \;=\; 0\,, \tag{10b}$$

$$K'(e'_{n+a}\,,\, e'_{n+b}) \;=\; -2K_{ab}\,. \tag{10c}$$

What can be deduced from that regarding the preservation of semi-simplicity and compactness of a Lie-algebra under this process of $\mathbb{C}$-extension? (A Lie-algebra is called compact, iff the Killing-form is negative definite.)

Problem 8

We consider once more the process of $\mathbb{C}$-extension of a real Lie-algebra L into another real Lie-algebra L' of twice the dimension, as explained in the previous exercise. We wish to show by way of example that two non-isomorphic Lie-algebras L_1 and L_2 may have isomorphic $\mathbb{C}$-extensions L'_1 and L'_2. Show that this is indeed the case for the two 3-dimensional Lie-algebras $L_{1,2} = \mathrm{Span}\{e_1, e_2, e_3\}$ with

$$[e_1, e_2] = e_3 \qquad [e_2, e_3] = e_1 \qquad [e_3, e_1] = e_2\,, \tag{11a}$$

$$[e_1, e_2] = e_3 \qquad [e_2, e_3] = e_1 \qquad [e_3, e_1] = -e_2\,. \tag{11b}$$

Note that (11a) is just the Lie-algebra of Problem 1, denoted by $\mathfrak{so}(3, \mathbb{R})$, whereas (11b) corresponds to the Lie-algebra of Problem 6, i.e. $\mathfrak{sl}(2, \mathbb{R})$, as one sees by a change of basis $e_1 = \frac{1}{2}(X^+ + X^-)$, $e_2 = \frac{1}{2}(X^- - X^+)$ and $e_2 = \frac{1}{2}H$ in (6). These two algebras are not isomorphic, as one sees by, e.g., looking that their Killing forms (one is negative definite, the other indefinite).

Let for $a = 1, 2, 3$ the six basis vectors for L'_1 und L'_2 be $e'_a = 1 \otimes e_a$, $e'_{3+a} = i \otimes e_a$, one sees that the basis transformation exchanging e'_1 with e'_4 and e'_3 with e'_6 transforms the relations L'_1 into that of L'_2.

The terminology is as follows: L is called a *real form* of L' if $L' = \mathbb{C} \otimes_{\mathbb{R}} L$. Hence we see that a given Lie-algebra may possess different (i.e. non-isomorphic) real forms, amongst them compact and non compact ones.

In the literature $\mathbb{C} \otimes_{\mathbb{R}} L$ is often called the "complexification" of L, because this real object possesses an obvious complex structure that may be used to turn it into a complex Lie-algebra. We deliberately do *not* follow this terminology.