Exercises for the Lecture on

Theory of Fundamental Interactions
(summer 2022)

by Domenico Giulini

Sheet 2

Problem 1

The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

1. Show that
$$\sigma_a \sigma_b = \delta_{ab} E_2 + i\, \varepsilon_{abc} \sigma^c , \tag{2}$$
where E_2 denotes the unit (2×2) matrix and where for notational consistency with the summation convention we write $\sigma^c := \delta^{cd}\sigma_d$ instead of σ_c.

2. Show that $\mathrm{Span}_{\mathbb{C}}\{E_2, \sigma_1, \sigma_2, \sigma_3\} = \mathrm{End}(\mathbb{C}^2)$.

3. Show the following identity that is valid for any $M \in \mathrm{End}(\mathbb{C}^2)$:
$$M = 2\,\mathrm{Trace}(M) E_2 - \sigma_a M \sigma^a . \tag{3}$$
We will use this identity in our discussion of $SL(2, \mathbb{C})$, the universal cover of the Lorentz-group.

Problem 2

In the lecture we discussed the existence of a decomposition of a semi-simple Lie-algebra into the Killing-orthogonal direct sum of simple ideals. Let now L be semi-simple and

$$L = \bigoplus_{a=1}^{N} I_n \tag{4}$$

such a decomposition. Show that it is unique (up to permutation of summands).

Hint: Let $I \subset L$ be a simple ideal; then $[I, L] := \mathrm{Span}\big\{[X, Y] : X \in I,\, Y \in L\big\} \subseteq I$. $[I, L]$ cannot be $\{0\}$ since this would imply that I is ..... in contradiction to the assumption that L is ..... Since $[I, L]$ is itself an ideal which is contained in I we have $[I, L] = I$. (4) implies $I = [I, L] = \oplus_a [I, I_a]$. Argue that simplicity of I implies that only one summand, say that for $a = i$, is not equal to $\{0\}$. Hence $I = [I, I_i]$, which, again by simplicity, immediately implies $I = I_i$.

Problem 3

Let $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ be the Pauli matrices (1) and $\vec{w} \in \mathbb{C}^3$.

1. Show that $\exp(\vec{w} \cdot \vec{\sigma})$ is an element of $\mathrm{SL}(2, \mathbb{C})$.
2. Expand $\exp(\vec{w} \cdot \vec{\sigma})$ and write the result as linear combination of $\{E_2, \vec{\sigma}\}$ so that all functions that occur have only real arguments.

Problem 4

Let L be a Lie-algebra and $[L, L] := \mathrm{Span}\big\{[X, Y] : X, Y \in L\big\}$.

1. Prove that $[L, L] \subseteq$ is an ideal.
2. Let L be simple (excluding one-dimensional L). Prove that L is perfect, i.e. $[L, L] = L$.
3. Let L be semi-simple. Prove that L is perfect. (Hint: Recall (4))
4. Let $L \subset \mathrm{End}(V)$ be a semi-simple Lie-algebra and $V \rtimes_\sigma L$ its semi-direct sum (compare Problem 5 of Sheet 1) with respect to the standard homomorphism $\sigma : L \to \mathrm{Der}(V)$, where $\sigma_X(v) := Xv$ (application of X to v). Under what conditions is $V \rtimes_\sigma L$ perfect?

Problem 5

We consider the Lie-algebra $\mathfrak{sl}(2, \mathbb{R})$ of the group $\mathrm{SL}(2, \mathbb{R})$ and again choose the basis

$$X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}$$

with

$$[X^+, X^-] = H, \quad [H, X^\pm] = \pm 2 X^\pm . \tag{6}$$

1. Let $\mathfrak{u}(n)$ denote the Lie-algebra of the group $U(n)$ of all unitary $n \times n$ matrices. $\mathfrak{u}(n)$ consists of all anti-hermitean $n \times n$ matrices: $X = -X^\dagger$.
2. Let $T : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{u}(n)$ be a homomorphism. Show that $\mathrm{Kernel}(T) = \mathfrak{sl}(2, \mathbb{R})$, i.e. T is trivial.
3. Show that this implies that $\mathrm{SL}(2, \mathbb{R})$ has no non-trivial finite-dimensional unitary representations. Show further that the corresponding statements also hold for $\mathfrak{sl}(2, \mathbb{C})$ and $\mathrm{SL}(2, \mathbb{C})$.

   Hint: We write for abbreviation $T(X) =: \hat{X}$. The anti-unitary matrix $\hat{X}^+$ satisfies $(\hat{X}^+)^2 = \frac{1}{2}\hat{X}^+ \cdot [\hat{H}, \hat{X}^+]$ (why?). Hence we have $\mathrm{Trace}\big((\hat{X}^+)^2\big) = 0$ (why?). From that you may follow $\hat{X}^+ = 0$ (how?). Simplicity of $\mathfrak{sl}(2, \mathbb{R})$ then yields $\mathrm{Kern}(T) = \mathfrak{sl}(2, \mathbb{R})$. Clearly, $\mathfrak{sl}(2, \mathbb{R})$ is a Lie-subalgebra of $\mathfrak{sl}(2, \mathbb{C})$. But the smallest ideal in $\mathfrak{sl}(2, \mathbb{C})$ containing $\mathfrak{sl}(2, \mathbb{R})$ (its ”Idealiser”) is $\mathfrak{sl}(2, \mathbb{C})$ itself (why?). Hence any Lie-homomorphismus from $\mathfrak{sl}(2, \mathbb{C})$ to $\mathfrak{u}(n)$ is trivial.

Problem 6

Let V be a real vector space with non-degenerate bilinear form $\omega : V \times V \to \mathbb{R}$, which we assume to be either symmetric ($\epsilon = 1$) or antisymmetric ($\epsilon = -1$); hence $\omega(v, w) = \epsilon\, \omega(w, v)$. We consider the group $O(V, \omega)$ of all ω-preserving linear maps (called the generalised orthogonal group of V with respect to ω):

$$O(V,\omega) := \Big\{A \in GL(V) : \omega(Av, Aw) = \omega(v, w)\ \forall v, w \in V\Big\}. \tag{7}$$

1. Show that $\mathrm{Lie}(O(V, \omega))$ is given by all ω-antisymmetric endomorphisms

$$\mathrm{Lie}\big(O(V,\omega)\big) := \Big\{X \in \mathrm{End}(V) : \omega(Xv, w) = -\omega(v, Xw)\ \forall v, w \in V\Big\}. \tag{8}$$

2. Let $\{e_a | a = 1, \cdots, n\}$ be a basis of V with respect to which we write for $\omega = V^* \otimes V^*$ and $X \in \mathrm{End}(V) \cong V \otimes V^*$ the the components $\omega_{ab} := \omega(e_a, e_b)$ and $X(e_b) = X^a{}_b e_a$. Show that $X \in \mathrm{Lie}\big(O(V, \omega)\big) \Leftrightarrow X_{ab} = -\epsilon\, X_{ba}$, where $X_{ab} := X^n{}_b \omega_{na}$.

3. Let $\{\theta^a | a = 1, \cdots, n\}$ be the basis of V^* dual to $\{e_a | a = 1, \cdots, n\}$ and define $\theta_a := \omega_{an}\theta^n$. Show that a basis for $\mathrm{Lie}\big(O(V, \omega)\big)$ is given by the

$$M_{ab} := e_a \otimes \theta_b - \epsilon e_b \otimes \theta_a \begin{cases} a < b & \text{for } \epsilon = 1\,, \\ a \le b & \text{for } \epsilon = -1\,. \end{cases} \tag{9}$$

   Note that the number of basis vectors is $n(n + 1)/2$ for $\epsilon = 1$ and $n(n + 1)/2$ for $\epsilon = -1$.

4. Show that the Lie-products (commutators) of the basis vectors are

$$\big[M_{ab}, M_{cd}\big] = \omega_{ad}M_{bc} + \omega_{bc}M_{ad} - \epsilon\omega_{ac}M_{bd} - \epsilon\omega_{bd}M_{ac}\,. \tag{10}$$

   Note that these cover the Lie-algebras for a large variety of groups, including the proper orthogonal groups and Lorentz groups in all dimensions (for $\epsilon = 1$) and symplectic groups in all dimensions (for $\epsilon = -1$). Note also that the right-hand side could have been written without explicit appearance of ϵ by writing $\epsilon\omega_{ac} = \omega_{ca}$ and $\epsilon\omega_{bd} = \omega_{db}$, but that would have somewhat destroyed the systematics of the index-permutations on the right-hand.

Problem 7

Let $G \subset GL(V)$ be a Lie-group. The corresponding "inhomogeneous group", denoted by IG, is defined to be $IG := V \rtimes_\alpha G$, i.e. its semi-direct product with the abelian group $(V, +)$ using the standard homomorphism $\alpha : G \to \mathrm{Aut}(V) \equiv GL(V)$, given by $\alpha_A(a) := Aa$ (application of A to a). Then $(a, A)(b, B) = (a + Ab\,,\ AB)$.

1. Let $s \mapsto \big(b(s), B(s)\big)$ be a differentiable curve through the identity, i.e. $a(0) = 0, A(0) = \_V$. An overdot denotes the derivative at $s = 0$. Then

$$\begin{aligned} \mathrm{Ad}_{(a,A)}(\dot b, \dot B) &:= \frac{d}{ds}\bigg|_{s=0} (a, A)\big(b(s)\,,\ B(s)\big)(a, A)^{-1} \\ &= \Big(A\dot b - A\dot B A^{-1}a\,,\ A\dot B A^{-1}\Big)\,. \end{aligned} \tag{11}$$

2. Let now also $t \mapsto \big(a(t), B(t)\big)$ be a differentiable curve through the identity. Then

$$\mathrm{ad}_{(\dot a,\dot A)}(\dot b,\dot B) := \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\big(a(t),A(t)\big)}\big(\dot b,\dot B\big) = \Big(\dot A\dot b - \dot B\dot a\,,\ [\dot A,\dot B]\Big)\,. \tag{12}$$

Problem 8

In this exercise we assume $G = GL(V)$. Then $\mathrm{Lie}(G) = \mathrm{End}(V) \cong V \otimes V^*$ and $\mathrm{Lie}(IG) = V \oplus (V \otimes V^*)$ as vector space. We also want to consider the dual vector space to $\mathrm{Lie}(IG)$, which we call $[\mathrm{Lie}(IG)]^*$. If $\{e_a \mid a = 1,\cdots,n\}$ and $\{\theta^a \mid a = 1,\cdots,n\}$ are dual bases of V and V^* respectively, we write $(y,Y) \in \mathrm{Lie}(IG)$ as $y = y^a e_a \in V$ and $Y = Y^a{}_b\, e_a \otimes \theta^b \in \mathrm{End}(V)$. Likewise, we write $(\sigma,\Sigma) \in \mathrm{Lie}(IG)$ as $\sigma = \sigma_a\theta^a$ and $\Sigma = \Sigma_a{}^b\theta^a \otimes e_b$. The action of (y,Y) under (σ,Σ) is then given by $(\sigma,\Sigma)[(y,Y)] = \sigma_a y^a + \Sigma_a{}^b Y^a{}_b$

From (11) we read off the *adjoint representation* Ad of the group IG on its own Lie-algebra $\mathrm{Lie}(IG)$: If $(a,A) \in IG$ and $(x,X) \in \mathrm{Lie}(IG)$ this is

$$\mathrm{Ad}_{(a,A)}(x,X) = \Big(Ax - \mathrm{Ad}_A(X)a\,,\ \mathrm{Ad}_A(X)\Big)\,, \tag{13}$$

As always, given a representation of a group on a vector space, the dual space carries the corresponding dual representation, given by the inverse-transposed:

$$\mathrm{Ad}^*_{(a,A)}(\sigma,\Sigma) := (\sigma,\Sigma) \circ \mathrm{Ad}_{(a,A)^{-1}}\,. \tag{14}$$

It is called the *co-adjoint representation* of IG.

1. Show that

$$\mathrm{Ad}^*_{(a,A)}(\sigma,\Sigma) := \Big(A^*\sigma\,,\ (A^* \otimes A)\Sigma + A^*\sigma \otimes a\Big)\,, \tag{15}$$

   where A^* is the dual (inverse transposed) action of A on V^*.

2. How does the corresponding formula read if G is a proper subgroup of $GL(V)$, e.g., as in the previous Problem, the subgroup leaving the bilinear form ω invariant?