Exercises for the Lecture on

Theory of Fundamental Interactions (summer 2022)

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Sheet 3

Problem 1

In the lecture we discussed the projection covering homomorphism $p : SU(2) \to SO(3)$ that satisfies

$$A(\vec{x}\cdot\vec{\sigma})A^\dagger = (R\vec{x})\cdot\vec{\sigma} \tag{1}$$

where σ_a are the Pauli-matrices. Let $\vec{n} \in \mathbb{R}^3$ be normalised.

Show that

$$A(\vec{n},\theta) := \exp\left(-\frac{i}{2}\theta\,\vec{n}\cdot\vec{\sigma}\right) \tag{2}$$

projects to a $SO(3)$-rotation by an angle θ in the oriented plane (i.e. the sign matters) whose normal is $\vec{n}$. (Hint: You might find it useful to decompose $\vec{x}$ into its parts parallel and perpendicular to $\vec{n}$.)

Problem 2

Show that *any* element $A \in SU(2)$ is of the form

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \tag{3}$$

where $a, b \in \mathbb{C}$ satisfying $|a|^2 + |b|^2 = 1$ and an overbar denotes complex conjugation. Argue that $SU(2)$ is connected and simply connected.

Problem 3

Let V be a real vector space and ω a non-degenerate bilinear form. We have seen that

$$O(V,\omega) := \left\{A \in GL(V) : \omega(Av, Aw) = \omega(v,w)\,, \forall v, w \in V\right\}, \tag{4}$$

$$\mathrm{Lie}\big(O(V,\omega)\big) := \left\{X \in \mathrm{End}(V) : \omega(Xv, w) + \omega(v, Xw) = 0\,, \forall v, w \in V\right\}. \tag{5}$$

There are two linear maps † and # from $\mathrm{End}(V)$ to itself, defined by

$$\omega(Xv, w) = \omega(v, X^\dagger w)\,, \tag{6}$$

$$\omega(v, Xw) = \omega(X^\# v, w)\,, \tag{7}$$

for all $v, w \in V$.

1. Show that each of these maps is the inverse of the other on all of $\mathrm{End}(V)$ and that on the subsets $O(V,\omega) \subset \mathrm{End}(V)$ and $\mathrm{Lie}\big(O(V,\omega)\big) \subset \mathrm{End}(V)$ they coincide and are hence involutions. If ω is symmetric or antisymmetric, the two maps coincide on all of $\mathrm{End}(V)$

2. Assuming that ω is symmetric or antisymmetric, show that the two linear maps $P_\pm : \mathrm{End}(V) \to \mathrm{End}(V)$

$$P_\pm(X) := \frac{1}{2}\big(X \pm X^\#\big) \tag{8}$$

   are idempotent, i.e. $P_\pm \circ P_\pm = \mathrm{id}_{\mathrm{End}(V)}$ and satisfy $P_\pm \circ P_\mp = 0$.

3. Let $\omega_\downarrow : V \to V^*$, $v \mapsto \omega(v,\cdot)$ be the usual isomorphism induced by ω and $\omega_\uparrow : V^* \to V$ its inverse. Show that if ω is either symmetric $\epsilon = 1$ or antisymmetric $\epsilon = -1$, then

$$\omega_\downarrow^\top = \epsilon\omega_\downarrow \quad \text{and} \quad \omega_\uparrow^\top = \epsilon\omega_\uparrow\,, \tag{9}$$

   where $\top$ indicates the usual transposed map. (Recall: If $f : V \to W$ is a linear map between vector spaces, then $f^\top : W^* \to V^*$, given by $f(\alpha) := \alpha \circ f$ is the transposed map. Note that only for $W = V^*$ are these maps between the same vector spaces.)

4. Show that, irrespectively of any symmetry assumptions on ω, we have

$$X^\# = \omega_\uparrow \circ X^\top \circ \omega_\downarrow\,, \tag{10}$$

   and hence

$$O(V,\omega) := \big\{A \in GL(V) : \omega_\downarrow \circ A \circ \omega_\uparrow = (A^{-1})^\top\big\}\,, \tag{11}$$

$$\mathrm{Lie}\big(O(V,\omega)\big) := \big\{X \in \mathrm{End}(V) : \omega_\downarrow \circ X \circ \omega_\uparrow = -X^\top\big\}\,. \tag{12}$$