Exercises for the Lecture on

Theory of Fundamental Interactions (summer 2022)

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Sheet 4

Problem 1

Like in the lecture, let $\sigma^0 = \tilde{\sigma}^0 = E_2$, $\sigma^a = -\tilde{\sigma}^a = a$-th Pauli matrix and $\sigma_\alpha = \eta_{\alpha\beta}\sigma^\beta$, $\tilde{\sigma}_\alpha = \eta_{\alpha\beta}\tilde{\sigma}^\beta$.

1. Show that for any $M \in \mathrm{End}(\mathbb{C}^2)$ (i.e. complex 2×2 matrix)
$$\sigma_\alpha M \tilde{\sigma}^\alpha = 2\,\mathrm{trace}(M)\, E_2\,. \tag{1}$$

2. Show that (indices in round brackets are symmetrised over, $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$)
$$\sigma_{(\alpha}\tilde{\sigma}_{\beta)} = E_2\,\eta_{\alpha\beta}\,. \tag{2}$$

3. Define a map $p : SL(2,\mathbb{C}) \to GL(\mathbb{R}^4)$ where $L := p(A)$ is implicitly defined by
$$A(x^\alpha\sigma_\alpha)A^\dagger = L^\alpha{}_\mu x^\mu \sigma_\alpha\,. \tag{3}$$
Show that
$$L^\alpha{}_\beta = \tfrac{1}{2}\mathrm{trace}(\tilde{\sigma}^\alpha A \sigma_\beta A^\dagger). \tag{4}$$

4. Show that p is a group homomorphism the image of which are the proper orthochronous Lorentz transformations and the kernel of which is $\{\pm E_2\}$.

5. Show that
$$A = p^{-1}(L) = \pm\frac{\sigma_\alpha L^\alpha{}_\beta\tilde{\sigma}^\beta}{\sqrt{\det\big(\sigma_\alpha L^\alpha{}_\beta\tilde{\sigma}^\beta\big)}} \tag{5}$$
for those L for which the determinant in the denominator on the right-hand side does not vanish.

Problem 2

Let η be a symmetric, non-degenerate bilinear form on the vector space V of dimension $n > 2$. We know that the Lie algebra of the corresponding inhomogeneous group $V \rtimes O(V,\eta)$ is given by $\mathrm{Lie}\big(V \rtimes O(V,\eta)\big) = \mathrm{Span}\{P_a, M_{ab} \mid 1 \leq a < b \leq n\}$, with

$$
\begin{aligned}
&[P_a, P_b] = 0\,, && (6a)\\
&[M_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b\,, && (6b)\\
&[M_{ab}, M_{cd}] = \eta_{ad}\, M_{bc} + \eta_{bc}\, M_{ad} - \eta_{ac}\, M_{bd} - \eta_{bd}\, M_{ac}\,. && (6c)
\end{aligned}
$$

1. Show explicitly (we already know this from general considerations) that this Lie-Algebra, which we call L, is perfect, i.e. $[L, L] = L$.
   Hint: Contract the right-hand of (6) *in an appropriate way with* η^{ab} *so that you can solve for the* P_a*'s and* M_{ab}*'s in terms of sums of Lie-products.*

2. Determine all one-dimensional representations of L and compare this with all one-dimensional representations of the translations only, which form an abelian ideal. What is remarkable here?

3. We consider a basis change of the form

$$
\begin{aligned}
P_a &\mapsto \bar{P}_a \;:=\; P_a\,, && (7a)\\
M_{ab} &\mapsto \bar{M}_{ab} \;:=\; M_{ab} - (X_a P_b - X_b P_a)\,, && (7b)
\end{aligned}
$$

   where X_a are the covariant components of a fixed vector in V, representing a translation. (This change of basis corresponds to a change of the origin o in affine space: $o \mapsto \bar{o} := o + X$.) Show that the Lie-producs of the new basis $(\bar{P}_a, \bar{M}_{ab})$ are identical to (6), i.e. are given by replacing (P_a, M_{ab}) with $(\bar{P}_a, \bar{M}_{ab})$ in (6).

Problem 3

The situation is just as in Problem 2. The group $G := V \rtimes O(V,\eta)$ acts on V as follows:

$$
\phi : G \times V \to V\,, \quad \big((a, A), v\big) \mapsto \phi_{(a,A)}(v) := Av + a\,. \tag{8}
$$

Let $C^\infty(V, \mathbb{F})$ be the infinite dimensional vector space of $\mathbb{F}$-valued (either $\mathbb{R}$ of $\mathbb{C}$) smooth functions on V. On $C^\infty(V, \mathbb{F})$ the group is represented via

$$
T : G \times C^\infty(V, \mathbb{F}) \to C^\infty(V, \mathbb{F})\,, \quad T_{(a,A)} f := f \circ \phi^{-1}_{(a,A)}\,. \tag{9}
$$

1. Show that this is indeed a representation

2. We are interested in the corresponding representation $\dot{T}$ of $\mathrm{Lie}(V \rtimes O(V,\eta))$ on $C^\infty(V, \mathbb{F})$. For that consider, as usual, a curve $s \mapsto (a(s), A(s))$ in G with $(a(0), A(0)) = (0, \mathrm{id}_V)$ and $d/ds|_{s=0}(a(s), A(s)) = (\dot{a}, \dot{A}) \in \mathrm{Lie}(V \rtimes O(V,\eta))$. Show that

$$
\dot{T}_{(\dot{a},\dot{A})} f(x) = Df(x)(-\dot{a} - \dot{A}x)\,. \tag{10}
$$

3. Let $\{e_a \mid a = 1, \cdots, n\}$ be a basis of V with dual basis $\{\theta^a \mid a = 1, \cdots, n\}$ of V^*. If we set $\theta_a := \eta_\downarrow(e_a) = \eta_{ab}\theta^b$, where $\eta_{ab} = \eta(e_a, e_b)$, then the $P_a := (e_a, 0)$ and $M_{ab} := (0, e_a \otimes \theta_b - e_b \otimes \theta_a)$ form a Basis of $\mathrm{Lie}(V \rtimes O(V, \eta))$ which just satisfies the relations (6). Show:

$$\dot{T}_{P_a} f = -\partial_a f\,, \quad \dot{T}_{M_{ab}} f = (x_a \partial_b - x_b \partial_a) f\,. \tag{11}$$

Here $x_a : V \to \mathbb{R}$, $x \mapsto \eta(x, e_a)$ are the co-variant coordinate functions with respect to the given basis. We can say that the images of the basis $\{P_a, M_{ab} : 1 \leq a < b \leq n\}$ of $\mathrm{Lie}(V \rtimes O(V, \eta))$ under $\dot{T}$ are the differential operators $-\partial_a$ and $(x_a \partial_b - x_b \partial_a)$, respectively. Check explicitly that they obey (6).

Problem 4

Consider again the group $O(V, \eta)$, where V is a 5-dimensional real vector space and η a symmetric, non-degenerate bilinear form of signature $(+, -, -, -, \sigma)$ ist, where $\sigma = \mp 1$. In case $\sigma = -1$ the group is also written $O(1, 4)$ and called *de Sitter*-group, in case $\sigma = +1$ it's written $O(2, 3)$ and called *anti-de Sitter*-group. A basis of the corresponding 10-dimensional Lie-algebra is $\{M_{ab} : 0 \leq a < b \leq 4\}$ which satisfies (6c).

We introduce the following renaming of the basis elements, where $a, b, c \in \{1, 2, 3\}$ und $\varepsilon_{abc} = \mathrm{sign}\begin{pmatrix}123\\abc\end{pmatrix}$, independent of whether the indices on ε are upstairs or downstairs (for the sake of obeying the summation convention):

$$D_a := \tfrac{1}{2}\varepsilon_a{}^{bc}\, M_{bc}\,, \tag{12a}$$
$$K_a := M_{0a}\,, \tag{12b}$$
$$T_a := M_{a4}\,, \tag{12c}$$
$$T_0 := M_{04}\,. \tag{12d}$$

1. Show that the Lie-algebras of $\mathrm{Lie}(O(1, 4))$ and $\mathrm{Lie}(O(2, 3))$ are characterised by the following 45 relations:

$$[D_a, D_b] = \varepsilon_{ab}{}^c\, D_c\,, \tag{13a}$$
$$[D_a, K_b] = \varepsilon_{ab}{}^c\, K_c\,, \tag{13b}$$
$$[D_a, T_b] = \varepsilon_{ab}{}^c\, T_c\,, \tag{13c}$$
$$[D_a, T_0] = 0\,, \tag{13d}$$
$$[K_a, K_b] = -\varepsilon_{ab}{}^c\, D_c\,, \tag{13e}$$
$$[K_a, T_b] = -\delta_{ab}\, T_0\,, \tag{13f}$$
$$[K_a, T_0] = -T_a\,, \tag{13g}$$
$$[T_a, T_b] = -\sigma\, \varepsilon_{ab}{}^c\, D_c\,, \tag{13h}$$
$$[T_a, T_0] = \sigma\, K_a\,. \tag{13i}$$

2. Show that the linear maps defined by

$$\Pi : (T_0, T_a, K_a, D_a) \mapsto (T_0, -T_a, -K_a, D_a)\,, \tag{14a}$$
$$\Theta : (T_0, T_a, K_a, D_a) \mapsto (-T_0, T_a, -K_a, D_a)\,, \tag{14b}$$
$$\Gamma : (T_0, T_a, K_a, D_a) \mapsto (-T_0, -T_a, K_a, D_a)\,, \tag{14c}$$

are Lie-automorphisms. How do you interpret them?

3. Show that the linear subspaces $U_1 := \mathrm{span}\{D_a, K_b : 1 \leq a, b \leq 3\}$ and $U_2 := \mathrm{span}\{D_a, T_b : 1 \leq a, b \leq 3\}$ are Lie-subalgebras. Perform the contractions over those. Do you know the Lie-algebras so obtained? What is their difference? What Lie-algebra is obtained if you perform these contractions one after the other? Does the result depend on the order? At what Lie-algebras do you arrive at if after the contractionen over either U_1 or U_2 you perform the contraction over $U_3 := \mathrm{span}\{D_a, T_0 : a = 1, 2, 3\}$. Again: do you know these Lie-algebras? What is their difference?

Problem 5

This Problem is directly connected with Problem 2 and tries to convey the conceptually important idea of what it takes to define "position observables" in special-relativistic theories.

Let $\mathrm{Lie}(V \rtimes O(V, \eta))$ be faithfully represented on the (infinite-dimensional) Lie-algebra $C^\infty(P, \mathbb{R})$ of real-valued smooth functions on the phase-space P of some mechanical system. The Lie-product on $C^\infty(P, \mathbb{R})$ is the Poisson bracket. Hence we have an embedding $\mathrm{Lie}(V \rtimes O(V, \eta)) \hookrightarrow C^\infty(P, \mathbb{R})$. Under this embedding the (P_a, M_{ab}) become real-valued functions on P obeying (6) if $[\cdot, \cdot]$ is replaced by the Poisson-bracket $\{\cdot, \cdot\}$. The values of these phase-space functions are the conserved quantities corresponding to the group of translations and Lorentz-transformations, i.e., energy-momentum and centre-of-mass and angular-momentum. Unlike in the original Lie-algebra $\mathrm{Lie}(V \rtimes O(V, \eta))$, there is an additional associative product in $C^\infty(P, \mathbb{R})$ given by pointwise multiplication. this allows us to form polynomials and broken rational functions from the phase-space functions (P_a, M_{ab}).

An "inertial system" (or "state of motion") is characterised by $n \in V$ with $\eta(n, n) = 1$. We now consider the phase-space functions defined by (7), which depend on $X \in V$, and ask for which values of X the four functions $\bar{M}_{ab} n^b$ have zeros (i.e. the corresponding conserved quantities vanish):

$$\bar{M}_{ab} n^b = 0\,. \tag{15}$$

Corresponding to (7b) this leads to the condition

$$\big((P_c n^c)\eta_{ab} - P_a n_b\big) X^b = M_{ab} n^b\,. \tag{16}$$

This is to be read as four conditions on the four real numbers X^b for each phase-space point. In what follows we restrict to those points on phase space where $P_c n^c \neq 0$ (which is the energy of the system in the inertial frame characterised by n).

Show that the general solution of (16) is then given by the one-parameter family (parameter λ):

$$X_a(\lambda) = \frac{1}{P_c n^c}\left(P_a\lambda + M_{ab}n^b\right). \tag{17}$$

The map $\lambda \mapsto X(\lambda)$ defines a wordline in spacetime which is sometimes called the "central-line" or the worldline of the "centre-of-mass". It is timelike if and only if P is timelike, i.e. $P_a P_b \eta^{ab} > 0$.

By construction, this worldline defines the set of all points in Minkowski space (here identified with V) at which the conserved quantity associated with boost symmetry vanishes. This worldline not only depends on the state of the system but also on the choice of the inertial system n. The latter dependence is a new feature in Special Relativity. It does not exist in classical mechanics (invariant under the inhomogeneous Galilei group), where the definitions of "centre-of-mass" with respect to inertial systems lead to a unique worldline in spacetime.

Hint: In order to derive (17) *use that the map Abbildung* $\Pi : V \to V$ *with Components* $\Pi^a_b = \delta^a_b - \frac{P^a n_b}{P_c n^c}$ *is just the projection of* V *onto* $n^\perp := \{v \in V : \eta(v, n) = 0\}$ *parallel to* $P \in V$ *(hence* not *die* orthogonal*projection onto* $n^\perp$*, which would be parallel to* n*).*

With respect to the solution worldline $X(\lambda)$ we can form $\bar{M}_{ab}$. Show that it is independent of λ. This $\bar{M}_{ab}$ is generally called the *spin tensor*. It is a phase-space function with values in $V^* \wedge V^*$) which depends on the inertial system (i.e. on n). We have

$$S_{ab} = M_{ab} + \frac{(P_a M_{bc} - P_b M_{ac})n^c}{n^c P_c} \tag{18}$$

Calculate the Poisson-brackets of the four phase-space functions $X^a(\lambda)$ defined through (17) amongst themselves and with P_a and M_{ab}. To what extent would you say that the X^a correspond to space-time "position observables"?