Exercises for the Lecture on

Theory of Fundamental Interactions
(summer 2022)

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Sheet 5

Problem 1

Consider the defining representation $D^{(1)}$ of $SO(3)$ on $\mathbb{R}^3$. Take its tensor product $D^{(1)} \otimes D^{(1)}$ on $\mathbb{R}^3 \otimes \mathbb{R}^3$ and show that the antisymmetric tensors, the symmetric-traceless tensors, and the tensors proportional to the euclidean metric (so-called "pure trace") form invariant subspaces of dimensions 3, 5, and 1, respectively. Are these subspaces irreducible?

Problem 2

Let $D^{(1)} : SO(3) \times V \to V$ be the defining representation of the rotation group on three-dimensional euclidean space $V = \mathbb{R}^3$. Let $V^{n\otimes} = V \otimes \cdots \otimes V$ be the n-fold tensor product, $V^{n\otimes}_{(s)}$ its total symmetrisation, and $V^{n\otimes}_{(s,t)}$ be the subspace in the latter of traceless tensors (the trace being defined via the euclidean inner product).

1. Prove that

$$\dim\bigl(V^{n\otimes}_{(s)}\bigr) = \frac{(n+2)!}{n!\,2} \quad \text{and} \quad \dim\bigl(V^{n\otimes}_{(s,t)}\bigr) = 2n+1 \tag{1}$$

2. Prove that the n-fold tensor product of the defining representation $D^{(1)}$ acts irreducibly on $V^{n\otimes}_{(s,t)}$. [Hint: First show that it linearly maps traceless symmetric tensors into traceless symmetric tensors. Then show that it is a representation of weight n and conclude irreducibility from the argument in the lecture.]

Problem 3

We consider $C^\infty(S^2, \mathbb{C})$, that is, the infinite-dimensional complex vector space of infinitely differentiable complex-valued functions on the 2-sphere $S^2 := \{\vec{x} \in \mathbb{R}^3 \mid \|\vec{x}| = 1\}$. The rotation group $SO(3)$ acts on S^2 via the defining representation on $\mathbb{R}^3$. A rotation about the oriented axis $\vec{n}$, where $\|\vec{n}\| = 1$, and angle α is given by

$$\begin{aligned} D(\vec{n},\alpha)\vec{x} &= \vec{x}_{\|} + \cos(\alpha)\,\vec{x}_{\perp} + \sin(\alpha)\,\vec{n} \times \vec{x}_{\perp} \\ &= \vec{x} + \bigl(\cos(\alpha) - 1\bigr)\,\vec{x}_{\perp} + \sin(\alpha)\,\vec{n} \times \vec{x} \end{aligned} \tag{2}$$

This action defines a linear representation of $SO(3)$ on $C^\infty(S^2, \mathbb{C})$ via

$$T(\vec{n}, \alpha)\psi = \psi \circ D^{-1}(\vec{n}, \alpha) = \psi \circ D(\vec{n}, -\alpha)\,. \tag{3}$$

The corresponding representation $\dot{T}$ of $\mathrm{Lie}(SO(3))$ on $C^\infty(S^2, \mathbb{C})$ is obtained by considering curves $D\big(\vec{n}(s), \alpha(s)\big)$ with $\vec{n}(0) = \vec{n}$, $\alpha(0) = 0$ and $\dot{\alpha}(0) = 1$.

1. Show that $\dot{T}$ is given by

$$\dot{T}_{\vec{n}}(\psi) = -\vec{n} \cdot (\vec{x} \times \vec{\nabla}\psi) \tag{4}$$

   and check that

$$\big[\dot{T}_{\vec{n}}, \dot{T}_{\vec{m}}\big] = \dot{T}_{\vec{n}\times\vec{m}}\,. \tag{5}$$

   Here $\vec{n}$ and $\vec{m}$ are considered as elements of $\mathrm{Lie}\big(SO(3)\big)$ with $\dot{D}(\vec{n})\vec{x} = \vec{n} \times \vec{x}$.

2. Let $\{\vec{e}_a \mid a = 1, 2, 3\}$ be an orthonormal Basis of $\mathbb{R}^3$. We define (compare Lecture)

$$J_a := i\,\dot{T}_{\vec{e}_a} \tag{6}$$

   so that $[J_a, J_b] = i\varepsilon_{ab}{}^{c}J_c$). As in the Lecture we form $J_\pm := J_1 \pm iJ_2$ and $J^2 := J_1^2 + J_2^2 + J_3^2$. Instead of $\vec{x}$ with $\|\vec{x}\| = 1$ we introduce spherical polar coordinates according to

$$x = \sin\theta\cos\varphi\,, \quad y = \sin\theta\sin\varphi\,, \quad z = \cos\theta\,. \tag{7}$$

   Show that

$$J_\pm = e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right), \tag{8a}$$

$$J_3 = -i\,\frac{\partial}{\partial\varphi}, \tag{8b}$$

$$J^2 = -\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right). \tag{8c}$$

3. Determine for each integer $\ell \in \mathbb{N}$ the irreducible subspace $V_\ell \subset C^\infty(S^2, \mathbb{C})$ by first solving the differential equation $J_+\psi_\ell = 0$ and then applying successively powers of J_-. The functions so obtained should be normalised in the L^2-norm on S^2 with respect to the measure $d\mu = \sin\theta\, d\theta \wedge d\varphi$. Show that this leads to the $2\ell + 1$ spherical harmonics $Y_{\ell m}$ for fixed ℓ and $-\ell \leq m \leq \ell$. These span the irreducible subspace of weight ℓ.

4. We know from general theory that $J^2 Y_{\ell m} = \ell(\ell + 1)Y_{\ell m}$. Check this using (8c). (It is sufficient to give the explicit formulae for the $Y_{\ell m}$ for $\ell = 1$.)

Problem 4

Let V be a two-dimensional complex vector space and $\bigvee^n V^* := \otimes^n_{(s)} V^*$ the n-fold totally symmetric tensor product of its dual space V^*.

Show that for any $\Theta \in \bigvee^n V^*$ there exist n elements $\phi^{(1)}, \cdots, \phi^{(n)} \in V^*$, so that $\Theta = \phi^{(1)} \vee \cdots \vee \phi^{(n)}$. Here $\vee$ denotes the symmetric tensor product. Further show that the $\phi^{(k)}$ are uniquely determined up to permutation and rescaling $\phi^{(k)} \mapsto \lambda^{(k)} \phi^{(k)}$, where the $\lambda^{(k)} \in \mathbb{C}$ obey $\lambda^{(1)} \cdot \lambda^{(2)} \cdots \lambda^{(n)} = 1$. The $\phi^{(k)}$ are called the *principal spinors* of Θ.

Hint. You may assume the following Lemma: Let $T \in \bigvee^n V^*$; then $T = 0$ if and only if $T(v, \cdots, v) = 0$ for all $v \in V$. (If you wish to prove the Lemma, consider $v = u + \lambda w$ for $\lambda \in \mathbb{C}$ and expand $T(v, \cdots, v) = 0$ in powers of λ. Conclude that, in particular, the coefficient of the linear term $\propto \lambda$ must vanish. This is true for any w and any u. Now proceed by iterating the argument for the remaining $n-1$ slots containing u....) Now, given the Lemma, it suffices to show that $\phi^{(1)}, \cdots, \phi^{(n)} \in V^*$ exist such that $\left(\Theta - \phi^{(1)} \vee \cdots \vee \phi^{(1)}\right)(v, v, \cdots, v) = 0$ for all $v \in V$. Expressing this in components (with respect to dual bases in V and V^* and their tensor products), this means

$$\left(\Theta_{A_1 \cdots A_n} - \phi^{(1)}_{(A_1} \cdots \phi^{(n)}_{A_n)}\right) v^{A_1} \cdots v^{A_n} = 0 \tag{9}$$

for all 2-tupel $(v^0, v^1) \in \mathbb{C}^2$. Without loss of generality we may assume $v^0 = 1$ (why?). Setting $v^1 =: z \in \mathbb{C}$ we get $\Theta(v, \cdots, v) = \Theta_{00\cdots 0} + z\, n \Theta_{10\cdots 0} + \cdots + z^n \Theta_{1\cdots 1}$. Using the fundamental theorem of algebra, conclude that there exist n complex tuples $\left(\phi^{(1)}_0, \phi^{(1)}_1\right), \cdots, \left(\phi^{(n)}_0, \phi^{(n)}_1\right)$ so that this polynomial in z equals $\left(\phi^{(1)}_0 + z\,\phi^{(1)}_1\right) \cdot \left(\phi^{(2)}_0 + z\,\phi^{(2)}_1\right) \cdots \left(\phi^{(n)}_0 + z\,\phi^{(n)}_1\right)$.

Problem 5

We recall that a *real structure* C on a complex vector space V is an antilinear involution (called "complex conjugation"); that is, an antilinear map $C : V \to V$ satisfying $C \circ C = \mathrm{id}_V$. A vector $v \in C$ is called *real* with respect to C if $C(v) = v$.

1. Show that a real structure is equivalent to a *linear* isomorphism $K : V \to \bar{V}$ satisfying $j^{-1} \circ K \circ j^{-1} = K^{-1}$. Here j is the natural anti-isomorphism $j : V \to \bar{V}$ defined in the Lecture.
2. Let V be a complex vector space and $\bar{V}$ its complex-conjugate vector space. Show that $V \otimes \bar{V}$ and $V \oplus \bar{V}$ carry *natural* (i.e. without specification of further structural elements) *real strucure*. Characterise the real vectors in each case.
3. Suppose now that V has a non-degenerate bilinear form ε. Show that now $V \oplus \bar{V}^*$ has a real structure (that now depends on ε). Characterise the real vectors.

Note: If V is the 2-dimensional complex vector space carrying the defining representation of $SL(2, \mathbb{C})$, then elements in V are called *Weyl-Spinors*, elements in $V \oplus \bar{V}^*$ *Dirac-Spinors*, and the real elements in $V \oplus \bar{V}^*$ *Majorana-Spinors* (after Ettore Majorana 1906 - ????, who mysteriously disappeared in 1938).

Problem 6

From the Lecture we recall the following: Let L be a real Lie-algebra and $\mathbb{C}\otimes_{\mathbb{R}}\mathsf{L}$, which is a real Lie-algebra of twice the dimension as L. It has a natural complex structure defined by $J_2(z\otimes X)=iz\otimes X$ (and real-linear extension). The reason why we put the index 2 on J_2 will become clear below. Hence we can make $\mathbb{C}\otimes\mathsf{L}$ into a complex Lie-algebra, which we now call $(\mathbb{C}\otimes\mathsf{L})^{\mathbb{C}}$, by defining $\mathbb{C}$-multiplication with $a+ib$ via $(a+ib)(z\otimes X):=[(a+ib)z]\otimes X$. The dimension of $(\mathbb{C}\otimes\mathsf{L})^{\mathbb{C}}$ over $\mathbb{C}$ equals the dimension of L over $\mathbb{R}$.

Now assume that L already comes equipped with a complex structure J, i.e. a (real-)linear map $J:\mathsf{L}\to\mathsf{L}$ satisfying $J\circ J=-\mathrm{id}_{\mathsf{L}}$ and $J([X,Y])=[J(X),Y]=[X,J(Y)]$. Let $\mathsf{L}^{\mathbb{C}}$ denote the complex Lie-algebra obtained from L by defining complex multiplication accordingly by $(a+ib)X:=aX+bJ(X)$ for all $a,b\in\mathbb{R}$ and all $X\in\mathsf{L}$. Note that the complex dimension of $\mathsf{L}^{\mathbb{C}}$ equals half the real dimension of L. (The origin of the complex structure J of L is not important here. It may stem from L having the form $\mathbb{C}\otimes\mathsf{L}'$ for some real Lie-algebra L', but we shall not need such an assumption.) Prove that

$$(\mathbb{C}\otimes\mathsf{L})^{\mathbb{C}}=\mathsf{L}^{\mathbb{C}}\oplus\bar{\mathsf{L}}^{\mathbb{C}}\,. \tag{10}$$

Here $\bar{\mathsf{L}}_{\mathbb{C}}$ denotes the complex-conjugate Lie-algebra to $\mathsf{L}_{\mathbb{C}}$, which is based on the complex-conjugate vector space on which the multiplication by $\mathbb{C}$ is defined by composition with complex-conjugation. Note that (10) means says $(\mathbb{C}\otimes\mathsf{L})^{\mathbb{C}}$ decomposes into two ideals of equal dimension if L has a complex structure.

Hint: Forst show that the $\mathbb{R}$-linear map J on L extends to the $\mathbb{R}$-linear map $J_1:=\mathrm{id}\otimes J$ on $\mathbb{C}\otimes\mathsf{L}$ and then also to a $\mathbb{C}$-linear map - also denoted by J_1 - to $(\mathbb{C}\otimes\mathsf{L})$ satisfying $J_1([X,Y])=[J_1(X),Y]=[X,J_1(Y)]$ and $J_1\circ J_1=-\mathrm{id}_{(\mathbb{C}\otimes\mathsf{L})^{\mathbb{C}}}$. Next consider on $(\mathbb{C}\otimes\mathsf{L})^{\mathbb{C}}$ the $\mathbb{C}$-linear maps

$$P_{\pm}:=\tfrac{1}{2}\Big(\mathrm{id}_{(\mathbb{C}\otimes\mathsf{L})^{\mathbb{C}}}\mp i\,J_1\Big)\,, \tag{11}$$

and show that they are projectors, i.e. satisfy $P_{\pm}\circ P_{\pm}=P_{\pm}$, $P_{\pm}\circ P_{\mp}=0$, and $P_{+}+P_{-}=\mathrm{id}_{(\mathbb{C}\otimes\mathsf{L})^{\mathbb{C}}}$ and also satisfy

$$\begin{aligned} &J_1\circ P_{\pm}=P_{\pm}\circ J_1=\pm\,i\,P_{\pm}\,,\\ &P_{\pm}([X,Y])=([P_{\pm}(X),Y])=[X,P_{\pm}(Y)]\,. \end{aligned} \tag{12}$$

These equations imply that $P_{\pm}$ project onto the eigenspaces of J_1 with eigenvalues $\pm i$ and that these eigenspaces are Lie-subalgebras.

Remark: The result of this exercise explains the result of the Lecture

$$\mathbb{C}\otimes\mathrm{Lie}(SL(2,\mathbb{C}))\cong[\mathbb{C}\otimes\mathrm{Lie}(SU(2))]\oplus[\mathbb{C}\otimes\mathrm{Lie}(SU(2))]\,, \tag{13}$$

i.e. that simplicity gets lost in the process of taking $\mathbb{C}\otimes(\cdots)$, while this is not true in other cases, like, e.g.,

$$\mathrm{Lie}(SL(2,\mathbb{C}))\cong\mathbb{C}\otimes\mathrm{Lie}(SU(2))\,. \tag{14}$$