Sheet 2: Solutions

Problem 1

1) $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Clearly $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = E_2$

$$\sigma_1 \cdot \sigma_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3$$

$$\sigma_2 \cdot \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_1$$

$$\sigma_3 \cdot \sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2$$

$$\Rightarrow \quad \sigma_a \cdot \sigma_b = \delta_{ab} E_2 + i\, \varepsilon_{ab}{}^{c} \sigma_c \qquad (2.1.1)$$

2) Have $\dim_{\mathbb{C}}(\mathrm{End}(\mathbb{C})) = 2 \times 2 = 4$.
$\rightsquigarrow$ it suffices to show that $\{E_2, \sigma_1, \sigma_2, \sigma_3\}$ are linearly independent. Now, if

$$\alpha E_2 + \beta^a \sigma_a = 0, \qquad (2.1.2)$$

taking the trace gives, because of

$$\mathrm{Trace}(\sigma_a) = 0, \quad 2\alpha = 0 \Leftrightarrow \alpha = 0. \qquad (2.1.3)$$

Multiplication of (2.1.2) with σ_b gives, using (2.1.1):

$$\alpha\sigma_b + \beta^a \delta_{ab} E_2 + i\beta^a \varepsilon_{ab}{}^{c} \sigma_c = 0 \qquad (2.1.4)$$

Taking the trace gives

$$2\beta_b = 0 \Leftrightarrow \beta^b = 0 \qquad (2.1.5)$$

As this is true for any $b \in \{1,2,3\}$, we get $\alpha = \vec{\beta} = 0 \Leftrightarrow \{E_2, \vec{\sigma}\}$ are linearly independent.

3) We write $E_2 =: \sigma^0$ (just a name) and set

$$M = M_\lambda \sigma^\lambda = \sum_{\lambda=0}^{3} M_\lambda \sigma^\lambda \qquad (2.1.6)$$

This we can do for any $M \in \mathrm{End}(\mathbb{C}^2)$ since $\{\sigma^0, \vec{\sigma}\}$ span $\mathrm{End}(\mathbb{C}^2)$, as just seen. Taking the trace gives

$$\mathrm{Trace}(M) = 2M_0 \quad , \text{ or}$$

$$M_0 = \frac{1}{2}\mathrm{Trace}(M). \qquad (2.1.7)$$

Now, using (2.1.1) we compute:

$$\sigma_a M \sigma^a = \sigma_a(M_0\sigma^0 + M_b\sigma^b)\sigma^a$$

$$= 3M_0\sigma^0 + M_b(\sigma_a[\delta^{ba}E_2 + i\varepsilon^{bac}\sigma_c])$$

$$= 3M_0\sigma^0 + M_b(\sigma^b + i\varepsilon^{bac}\, i\,\varepsilon_{acd}\sigma^d)$$

$$= 3M_0\sigma^0 - M_b\sigma^b = 4M_0\sigma^0 - M$$

$$\Rightarrow M = 2\,\mathrm{Trace}(M)E_2 - \sigma_a M\sigma^a \qquad (2.1.8)$$

Problem 2

$$L = \bigoplus_{a=1}^{N} I_a \tag{2.2.1}$$

L = semi-simple (no non-trivial abelian ideal)

I_a = simple ideals

We wish to show that any simple ideal $I \subseteq L$ is one of the I's.

So let's assume $I \subset L$ is a simple ideal, then

$$[I, L] := \text{Span}\{[X, Y] : X \in I, Y \in L\} \subseteq I \tag{2.2.2}$$

cannot be $\{0\}$ because then

$$[I, I] \subseteq [I, L] = \{0\} \tag{2.2.3}$$

and I would be an abelian ideal of L, in contradiction to L being semi simple. Hence $[I, L] \neq \{0\} \subseteq I$ is itself an ideal of $I \Rightarrow [I, L] = I$

From (2.2.1)

$$I = [I, L] = \bigoplus_{a=1}^{N} [I, I_a]$$

as I is simple only one summand eg for $a = i$, can be non zero $\Rightarrow I = [I, I_i]$ = ideal in I_i (simple) $\Rightarrow I = I_i$.

Problem 3

1.) $\exp(\vec{w}\cdot\vec{\sigma})$, $\vec{w} \in \mathbb{C}^3$

$$\det \exp(\vec{w}\cdot\vec{\sigma}) = \exp(\underbrace{\mathrm{Trace}(\vec{w}\cdot\vec{\sigma})}_{=0})$$
$$= 1$$

$$\Rightarrow \exp(\vec{w}\cdot\vec{\sigma}) \in SL(2,\mathbb{C})$$

2.)
$$\exp(\vec{w}\cdot\vec{\sigma}) := \sum_{n=0}^{\infty} \frac{(\vec{w}\cdot\vec{\sigma})^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(\vec{w}\cdot\vec{\sigma})^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(\vec{w}\cdot\vec{\sigma})^{2n+1}}{(2n+1)!}$$

From (2.1.1) we know
$$(\vec{w}\cdot\vec{\sigma})^2 = \vec{w}^2 E_2 \qquad (\vec{w}^2 := \vec{w}\cdot\vec{w})$$

For $\vec{w} \in \mathbb{C}^2 \setminus \{0\}$ $\vec{w}$ may be zero, > 0 or < 0.

1 Case : $\vec{w}^2 = 0$

$$\Rightarrow \exp(\vec{w}\cdot\vec{\sigma}) = E_2 + \vec{w}\cdot\vec{\sigma}$$
$$= E_2 + \vec{b}\cdot\vec{\sigma} + i\vec{r}\vec{\sigma}$$

where $\vec{w} = \vec{b} + i\vec{r}$, $\vec{b}, \vec{r} \in \mathbb{R}^3$

2. Case $\vec{W}^2 = \theta^2 \in \mathbb{C} \setminus \{0\}$.

$$(\vec{W}\cdot\vec{\sigma})^2 = \vec{W}^2 E_2 = \theta^2 E_2$$

$$\exp(\vec{W}\cdot\vec{\sigma}) = \sum_{n=0}^{\infty} \frac{(\vec{W}\cdot\vec{\sigma})^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(\vec{W}\cdot\vec{\sigma})(\vec{W}\cdot\vec{\sigma})^{2n}}{(2n+1)!}$$

$$= E_2 \sum \frac{\theta^{2n}}{(2n)!} + \vec{W}\cdot\vec{\sigma} \sum_{n=0}^{\infty} \frac{\theta^{2n}}{(2n+1)!}$$

$$= E_2 \cosh(\theta) + \frac{\vec{W}}{\theta}\cdot\vec{\sigma}\, \sinh(\theta)$$

Subcase 2.1: $\vec{W}^2 = \theta^2 = -\alpha^2$, $\alpha \in \mathbb{R}_+$

$\curvearrowright$ $\theta = i\alpha$ (we shall choose that root)

$$\Rightarrow \exp(\vec{W}\cdot\vec{\sigma}) = \cos(\alpha) E_2 + i\vec{n}\cdot\vec{\sigma} \sin(\alpha)$$

where $\vec{n} := \frac{\vec{W}}{\theta}$, $\vec{n}^2 = 1$.

Subcase 2.2: $\vec{W}^2 = \theta^2 = +\alpha^2$, $\alpha \in \mathbb{R}$

$$\Rightarrow \exp(\vec{W}\cdot\vec{\sigma}) = \cosh(\alpha) E_2 + \vec{n}\cdot\vec{\sigma} \sinh(\alpha)$$

In general $\vec{W} = \vec{b} + i\vec{r}$

$$\vec{W}^2 = \|\vec{b}\|^2 - \|\vec{r}\|^2 + 2i\,\vec{b}\cdot\vec{r}$$

This is a mixture of a boost in $\vec{b}$-direction and a rotation around the $\vec{r}$ axis, as we shall see in the Lecture.

Problem 4

1)

$$[L, L] := \mathrm{Span}\{[X, y] : X, y \in L\} \qquad (2.4.1)$$

is obviously a linear subspace that inherits the $[\cdot,\cdot]$ - product under which it is closed: $X, y \in [L, L] \Rightarrow [X, y] \in [L, L]$ (tautologically). The properties of $[\cdot,\cdot]$ restricted to $[L, L]$ follow from that of $[\cdot,\cdot]$.

2) If L is simple, since $[L, L]$ is an ideal, $[L, L] = \{0\}$ or $[L, L] = L$.
If $\dim(L) \geq 2$ $[L, L] = \{0\}$ is impossible, for otherwise any 1-dim. subspace of L would be a non-trivial ideal. Hence $[L, L] = L$.

3) From
$$L = \bigoplus_{a=1}^{N} I_a \qquad (2.4.2)$$

$$\Rightarrow [L, L] = \bigoplus_{a=1}^{N} [I_a, I_a]$$

$$= \bigoplus I_a \qquad (2.4.3)$$

since $[I_a, I_a] = I_a$ by 2)

4) In a semi-direct product $V \rtimes G$, with $G \subseteq \mathrm{End}(V)$ we have

$$\mathrm{Lie}(V \rtimes G) \cong V \times \mathrm{Lie}(G) \tag{2.4.4}$$

with

$$[(\dot{a}, \dot{A}), (\dot{b}, \dot{B})] = (\dot{A}\cdot\dot{b} - \dot{B}\cdot\dot{a}, [\dot{A}, \dot{B}]) \tag{2.4.5}$$

If G is semisimple and hence perfect, sums of terms $[A, B]$ give all of $\mathrm{Lie}(G)$. For $\mathrm{Lie}(V \rtimes G)$ to be perfect sums of terms $A \cdot b$, $A \in G$, $b \in V$ must give all of V. This is true if $G \subseteq \mathrm{End}(V)$ acts irreducibly on V, ie.

$$\mathrm{Span}\{A \cdot b : A \in G, b \in V\} = V \tag{2.4.6}$$

Problem 5

1) $$U(n) = \{ A \in GL(\mathbb{C}^n) : \langle AX, Ay \rangle \; \forall X, y \in \mathbb{C}^n \} \qquad (2.5.1)$$

Let $s \mapsto A(s)$ be C^1-curve in $U(n)$ such that $A(0) = \mathrm{id}$,

$$\langle A(s)X, A(s)y \rangle = \langle X, y \rangle \qquad (2.5.2)$$

$\frac{d}{ds}\Big|_{s=0}$ of (2.5.2) gives

$$\langle \dot{A}X, y \rangle + \langle X, \dot{A}y \rangle = 0 \qquad \forall X, y$$

$$\Leftrightarrow \quad \dot{A}^{+} = -\dot{A}$$

$$\Rightarrow \mathrm{Lie}(U(n)) = \{ X \in \mathrm{End}(\mathbb{C}^n) : X^{+} = -X \} \qquad (2.5.3)$$

2) $$T : \mathrm{Lie}(SL(2,\mathbb{R})) \to \mathrm{Lie}(U(n)) \qquad (2.5.4)$$

Lie-homomorphism

$$\left.\begin{array}{l} \text{We write } T(X^{\pm}) =: \hat{X}^{\pm} \text{ and} \\ T(H) = \hat{H} \text{; then} \end{array}\right\} \qquad (2.5.5)$$

$T([H, X^{+}]) = 2\,T(X^{+})$ implies

$$\hat{H}\hat{X}^{+} - \hat{X}^{+}\hat{H} = 2\hat{X}^{+}$$

$$\Rightarrow (\hat{X}^{+})^2 = \frac{1}{2}\hat{X}^{+}[\hat{H}, \hat{X}^{+}]$$

$$= \frac{1}{2}\left(\hat{X}^{+}\hat{H}\hat{X}^{+} - (\hat{X}^{+})^2 H\right) \qquad (2.5.6)$$

Hence

$$\mathrm{Trace}\left[(\hat{X}^+)^2\right]$$

$$= \frac{1}{2}\ \mathrm{Trace}\left(\hat{X}^+\hat{H}\hat{X}^+ - \hat{X}^+\hat{X}^+H\right) = 0$$

cyclic property of trace

Hence $\mathrm{Trace}\,(\hat{X}^+)^2 = 0$ (2.5.7)

But since $\hat{X}^+$ is anti-hermitean

$$\mathrm{Trace}\,(\hat{X}^+)^2 = -\sum_{a,b=1}^{n} |\hat{X}^+_{ab}|^2 \leq 0 \qquad (2.5.8)$$

and $= 0 \iff \hat{X}^+ = 0$

Hence $X^+ \in \mathrm{Kernel}\,(T)$. (2.5.9)

As Kernel $T \subseteq \mathrm{Lie}\,(SL(2,\mathbb{R}))$ is an ideal, which, as just seen, contains X^+, it must be identical to all of $\mathrm{Lie}\,(SL(2,\mathbb{R}))$ by the fact that $\mathrm{Lie}(SL(2,\mathbb{R}))$ is simple (Sheet 1, Problem 6).

$$\left.\begin{aligned} &\Rightarrow \mathrm{Kernel}\,(T) = \mathrm{Lie}\,(SL(2,\mathbb{R})) \\ &\Leftrightarrow T = 0 \quad \text{(the trivial map with image 0).} \end{aligned}\right\} \qquad (2.5.10)$$

3) Let

$$D : SL(2,\mathbb{R}) \to U(n) \tag{2.5.11}$$

be a group homomorphism (a unitary representation). Then

$$D_* : \mathrm{Lie}(SL(2,\mathbb{R})) \to \mathrm{Lie}(U(n)) \tag{2.5.12}$$

is a homomorphism of Lie-algebras, where D_* is the differential of D at the group indentity. But we have just seen that $D_* = 0$. Now, we have

$$\frac{d}{ds}\Big|_{s=0} D(\exp(sX)) = D_*(X) = 0 \tag{2.5.13}$$

$\Rightarrow$ D maps all elements in the image of the exponential map to the same element in $U(n)$, i.e. the identity.

As $SL(2,\mathbb{R})$ is connected, and any element of a connected Lie group is the finite product of elements in the image of exp, we have

$$D(A) = D(\exp(X_1) \cdots \exp(X_n))$$
$$= D(\exp(X_1)) \cdots D(\exp(X_n)) = e \tag{2.5.14}$$

$\Rightarrow$ D is trivial. (Same for $SL(2,\mathbb{C})$).

Problem 6

$$O(V,\omega) := \{A \in GL(V) : \omega(Av, Aw) = \omega(v,w),\ \forall v,w \in V\} \quad (2.6.1)$$

1.) Let $s \mapsto A(s)$ be C^1-curve in $O(V,\omega)$ with $A(0) = e$, then

$$\omega(A(s)v, A(s)w) = \omega(v,w) \quad (2.6.2)$$

$\frac{d}{ds}\Big|_{s=0}$ of that gives

$$\omega(\dot{A}v, w) + \omega(v, \dot{A}w) = 0 \quad (2.6.3)$$

hence

$$\mathrm{Lie}(O(V,\omega)) = \{X \in \mathrm{End}(V) : \omega(Xv, w) = -\omega(v, Xw),\ \forall v,w \in V\} \quad (2.6.4)$$

2) $\{e_a \mid a = 1, \dots, n\}$, $\omega_{ab} := \omega(e_a, e_b)$ then

$$\omega(Xe_a, e_b) = -\omega(e_a, Xe_b) \quad (2.6.5)$$

$$\Leftrightarrow \quad X^c{}_a\, \omega_{cb} = -X^c{}_b\, \omega_{ac} \quad (2.6.6)$$

We set
$$\omega(v,w) = \varepsilon\, \omega(w,v) \quad (2.6.7)$$

where $\varepsilon = \pm 1$ $\quad (2.6.8)$

and also define

$$X^c{}_a\, \omega_{cb} =: X_{ba}. \quad (2.6.9)$$

Then (2.6.6) is equivalent to

$$X_{ba} = - X^{c}{}_{b} \, \omega_{ac}$$

$$= - \varepsilon X^{c}{}_{b} \, \omega_{ca}$$

$$= - \varepsilon X_{ab} \qquad (2.6.10)$$

or

$$X_{ab} = - \varepsilon X_{ba} \qquad (2.6.11)$$

3) $\{ \theta^a \mid a = 1, \cdots, n \}$ dual basis to $\{ e_a \mid a = 1, \cdots, n \}$, s.t.

$$\theta^a (e_b) = \delta^a{}_b \qquad (2.6.12)$$

$$M_{ab} := e_a \otimes \theta_b - \varepsilon \, e_b \otimes \theta_a$$

with $\theta_a := \omega_{ab} \theta^b$. (2.6.13)

Have
$$\left.\begin{aligned} M_{ab} &= - M_{ba} \quad \text{for } \varepsilon = +1 \\ &= + M_{ba} \quad \text{for } \varepsilon = -1 \end{aligned}\right\} (2.6.14)$$

i.e. $\frac{1}{2} n(n-1)$ vectors for $\varepsilon = +1$ and $\frac{1}{2} n(n+1)$ vectors for $\varepsilon = -1$.

They are linearly independent; (or let $e^c = \omega^{ca} e_a$ with $\omega^{ac} \omega_{bc} = \delta^a{}_b$, then $\theta_a (e^b) = \omega_{ac} \theta^c (\omega^{bd} e_d) = \delta_a{}^b$. Hence, if for $X^{ab} \in \mathbb{R}$, $X^{ab} = -\varepsilon X^{ba}$,

$$\sum_{a,b} X^{ab} M_{ab} = 0 \qquad (2.6.15)$$

application to e^c

$$\sum (X^{ab} M_{ab})(e^c)$$

$$= \sum X^{ab} (e_a \delta_b^c - \varepsilon e_b \delta_a^c)$$

$$= X^{ac} e_a - \varepsilon X^{cb} e_b = 2 X^{ac} e_a$$

$$= 0 \iff X^{ac} = 0 \text{ for all } a \quad (2.6.16)$$

and the given c. But since the argument can be repeated for any c we get $X^{ab} = 0 \quad \forall\, a, b \in \{1, \dots, n\}$.
$\Rightarrow \{M_{ab}\}$ are linearly independent and of right number $\Rightarrow$ basis.

It remains to show that $M_{ab} \in$ Lie $(O(V, \omega))$. Hence we need to show that

$$\omega(M_{ab} e_c, e_d) = -\omega(e_c, M_{ab} e_d) \quad (2.6.17)$$

The left-hand side is

$$\omega(e_a \omega_{bc} - \varepsilon e_b \omega_{ac}, e_d)$$

$$= \omega_{ad} \omega_{bc} - \varepsilon \omega_{ac} \omega_{bd}$$

The right hand side is

$$-[\omega(e_c, e_a \omega_{bd} - \varepsilon e_b \omega_{ad})]$$

$$= -[\omega_{ca} \omega_{bd} - \varepsilon \omega_{cb} \omega_{ad}]$$

$$= -\varepsilon W_{ac} W_{bd} + W_{bc} W_{ad} \qquad (2.6.18)$$

Hence both sides coincide.

4.)

$$[M_{ab}, M_{cd}]$$

$$= (e_a \otimes \theta_b - \varepsilon\, e_b \otimes \theta_a)(e_c \otimes \theta_d - \varepsilon\, e_d \otimes \theta_c)$$

$$- (a \leftrightarrow c)(b \leftrightarrow d)$$

$$= \underbrace{W_{bc}\, e_a \otimes \theta_d}_{1} + \underbrace{W_{ad}\, e_b \otimes \theta_c}_{2}$$

$$- \underbrace{\varepsilon W_{ac}\, e_b \otimes \theta_d}_{3} - \underbrace{\varepsilon W_{bd}\, e_a \otimes \theta_c}_{4}$$

$$- \underbrace{W_{da}\, e_c \otimes \theta_b}_{2} - \underbrace{W_{cb}\, e_d \otimes \theta_a}_{1}$$

$$+ \underbrace{\varepsilon W_{ca}\, e_d \otimes \theta_b}_{3} + \underbrace{\varepsilon W_{db}\, e_c \otimes \theta_a}_{4}$$

$$= W_{bc}(e_a \otimes \theta_d - \varepsilon\, e_d \otimes \theta_a)$$

$$+ W_{ad}(e_b \otimes \theta_c - \varepsilon\, e_c \otimes \theta_b)$$

$$- \varepsilon W_{ac}(e_b \otimes \theta_d - \varepsilon\, e_d \otimes \theta_b)$$

$$- \varepsilon W_{bd}(e_a \otimes \theta_c - \varepsilon\, e_c \otimes \theta_a)$$

$$= W_{bc} M_{ad} + W_{ad} M_{bc}$$

$$- \varepsilon W_{ac} M_{bd} - \varepsilon W_{bd} M_{ac} \qquad (2.6.19)$$

Problem 7

Consider $G \subseteq GL(V)$ and

$$IG := V \rtimes G \qquad (2.7.1)$$

s.t. $$(a, A)(b, B) = (a + Ab, AB); \qquad (2.7.2)$$

then $$(a, A)^{-1} = (-A^{-1}a, A^{-1}) \qquad (2.7.3)$$

1.) $(a, A)(b, B)(a, A)^{-1}$

$$= (a + Ab, AB)(-A^{-1}a, A^{-1})$$

$$= (a + Ab - ABA^{-1}a, ABA^{-1}). \qquad (2.7.4)$$

If b and B depend on parameter s (a and A are constant), then $\frac{d}{ds}\Big|_{s=0}$ leads to

$$Ad_{(a,A)}(\dot{b}, \dot{B}) = (A\dot{b} - A\dot{B}A^{-1}a, A\dot{B}A)$$

$$= (A\dot{b} - Ad_A(\dot{B})(a), Ad_A(\dot{B})) \qquad (2.7.5)$$

This is the adjoint representation of IG on its Lie-algebra, Lie(IG); see below. If we now consider a curve $t \mapsto (a(t), A(t))$ with $a(0) = 0$, $A(0) = e$ (note: 0 is the identity in the abelian group V), we can

take the derivative $\frac{d}{dt}\big|_{t=0}$ of (2.7.5).
Note that

$$\frac{d}{dt}\Big|_{t=0} A\dot{B}A^{-1}a = \dot{B}\dot{a} \qquad (2.7.6)$$

(all other terms vanish due to $a(0)=0$),
so that

$$\mathrm{ad}(\dot{a},\dot{A})(\dot{b},\dot{B}) = \left(\dot{A}\dot{b} - \dot{B}\dot{a}, [\dot{A},\dot{B}]\right). \qquad (2.7.7)$$

This is what we already used in
(2.4.5).

Problem 8

From (2.7.5) we read off, setting $b = x$ and $\dot{B} = X$,

$$\mathrm{Ad}_{(a,A)}(x, X) = \left(Ax - \mathrm{Ad}_A(X)\,a\,,\ \mathrm{Ad}_A(X)\right) \qquad (2.8.1)$$

If $(\sigma, \Sigma) \in \mathrm{Lie}(IG)$, so that

$$(\sigma, \Sigma)(y, y) = \sigma(y) + \Sigma(y)$$

$$= \sigma_a y^a + \Sigma_a{}^b y^a{}_b \qquad (2.8.2)$$

then we can compute

$$\left\{\mathrm{Ad}^*_{(a,A)}(\sigma, \Sigma)\right\}(x, X)$$

$$:= (\sigma, \Sigma)\left(\mathrm{Ad}_{(a,A)^{-1}}(x, X)\right)$$

$$= (\sigma, \Sigma)\ \mathrm{Ad}_{(-A^{-1}a,\, A^{-1})}(x, X)$$

$$= (\sigma, \Sigma)\left(A^{-1}x - \mathrm{Ad}_{A^{-1}}(X)(-A^{-1}a),\ \mathrm{Ad}_{A^{-1}}(X)\right)$$

$$= (\sigma, \Sigma)\left(A^{-1}x + A^{-1}Xa,\ A^{-1}XA\right)$$

$$= \sigma(A^{-1}x) + \sigma(A^{-1}Xa) + \Sigma(A^{-1}XA)$$

$$= \sigma'(x) + \Sigma'(X) \qquad (2.8.3)$$

$$\Leftrightarrow \quad \sigma' = A^*\sigma \quad (:= \sigma \circ A^{-1}) \qquad (2.8.4)$$

$$\Sigma' = \mathrm{Ad}^*_A \Sigma + A^*\sigma \otimes a \qquad (2.8.5)$$

In case G is a subgroup of $GL(V)$ the Lie-algebra $\mathrm{Lie}(G)$ is a linear subspace of $\mathrm{End}(V)$. The element X in (2.8.3) is then from

$$X \in \mathrm{Lie}(G) \subset \mathrm{End}$$

and

$$\Sigma'(X) = \Sigma(A^{-1} X A) + \sigma(A^{-1} X a) \qquad (2.8.6)$$

for all $X \in \mathrm{Lie}(G)$.

If $$P : \mathrm{End}(V) \to \mathrm{Lie}(G) \qquad (2.8.7)$$

is a projection map, then (2.8.6) implies

$$\Sigma' = \overset{*}{\mathrm{Ad}}_A \Sigma + \underbrace{P^*(A^* \sigma \otimes a)}_{(A^* \sigma \otimes a) \circ P} \qquad (2.8.8)$$

For example, if $G = O(V, \omega)$, so that

$$\mathrm{Lie}(G) = \{ X \in \mathrm{End}(V) : \omega(Xv, w) + \omega(v, Xw) = 0 \quad \forall v, w \in V \} \qquad (2.8.9)$$

with $$\omega(v, w) = \varepsilon\, \omega(w, v) \qquad (2.8.10)$$

and $$\omega(v, \cdot) = \omega_\downarrow(v) \qquad (2.8.11)$$
(index lowering),

have $$\omega_\downarrow(v)(w) = v(\omega_\downarrow^T w) = \varepsilon\, \omega_\downarrow(w)(v) = \varepsilon\, v(\omega_\downarrow(w)) \qquad (2.8.12)$$

Hence

$$\omega\downarrow : V \to V^*, \qquad (2.8.13)$$

$$\omega\uparrow := (\omega\downarrow)^{-1} : V^* \to V \qquad (2.8.14)$$

$$\omega\downarrow^T : V^{**} \equiv V \to V^* \qquad (2.8.15)$$

with

$$\omega\downarrow^T = \varepsilon\, \omega\downarrow \qquad (2.8.16)$$

$$\omega\uparrow^T = \varepsilon\, \omega\uparrow \qquad (2.8.17)$$

Then

$$\omega(Xv, w) + \omega(v, Xw) = 0$$

$$\Leftrightarrow \quad \omega\downarrow \circ X(v)(w) + \omega\downarrow(v)(Xw) = 0$$

$$\Leftrightarrow \quad \omega\downarrow \circ X + X^T \circ \omega\downarrow = 0$$

$$\Leftrightarrow \quad X + \omega\uparrow \circ X^T \circ \omega\downarrow = 0 \qquad (2.8.18)$$

This is the linear relation $X \in \mathrm{End}(V)$ has to satisfy in order to be in $\mathrm{Lie}(G)$.

Consider the two linear maps

$$P_\pm : \mathrm{End}(V) \to \mathrm{End}(V)$$

$$X \mapsto P_\pm(X) := \frac{1}{2}(X \pm \omega\uparrow \circ X^T \circ \omega\downarrow) \qquad (2.8.19)$$

Have

$$P_+ + P_- = \mathrm{id}|_{\mathrm{End}(V)} \tag{2.8.20}$$

$$P_\pm P_\pm (X) = P_\pm^2 (X)$$

$$= P_\pm \left(\tfrac{1}{2} (X \pm \omega_\uparrow \circ X^T \circ \omega_\downarrow) \right)$$

$$= \tfrac{1}{4} [X \pm \omega_\uparrow \circ X^T \circ \omega_\downarrow$$

$$\pm \omega_\uparrow (X \pm \omega_\uparrow \circ X^T \circ \omega_\downarrow)^T \circ \omega_\downarrow]$$

$$= \tfrac{1}{4} [X \pm 2 \, \omega_\uparrow \circ X^T \circ \omega_\downarrow$$

$$+ \underbrace{\omega_\uparrow (\underset{\varepsilon \omega_\downarrow}{\underset{\parallel}{\omega_\downarrow^T}} \circ X \circ \underset{\varepsilon \omega_\uparrow}{\underset{\parallel}{\omega_\uparrow^T}}) \circ \omega_\downarrow}_{X}]$$

$$= P_\pm (X) \tag{2.8.21}$$

Similarly

$$P_+ \circ P_- (X) = P_- \circ P_+ (X) = 0 \tag{2.8.22}$$

Hence $P_\pm$ are projection operators and

$$\mathrm{Lie}(G) = \mathrm{Ker}(P_+) = \mathrm{Image}(P_-) \tag{2.8.23}$$

Applied to (2.8.8) with $P = P_-$ we get

$$\mathrm{Ad}^*_{(a,A)}(\sigma, \Sigma) = (\sigma', \Sigma') \tag{2.8.24}$$

where

$$\sigma' = A^* \sigma \tag{2.8.25}$$

$$\Sigma' = \mathrm{Ad}^*_A \Sigma + P_-^*(A^*\sigma \otimes a) \tag{2.8.26}$$

For example, for the ordinary orthogonal group with respect to euclidean metric P_- is just the antisymmetrisation in the tensor product.