Sheet 4 : Solutions

Problem 1

$$\sigma^{\alpha} = (E_2, \vec{\sigma}) \tag{4.1.1}$$

$$\tilde{\sigma}^{\alpha} = (E_2, -\vec{\sigma}) \tag{4.1.2}$$

$$\sigma_{\alpha} := \eta_{\alpha\beta}\,\sigma^{\beta} = (E_2, -\vec{\sigma}) \tag{4.1.3}$$

$$\tilde{\sigma}_{\alpha} := \eta_{\alpha\beta}\,\tilde{\sigma}^{\alpha} = (E_2, \vec{\sigma}) \tag{4.1.4}$$

1)

We know from Problem 1 of Sheet 2 that

$$M = 2\,\mathrm{Trace}(M)\,E_2 - \underbrace{\sigma_a M \sigma^a}_{\delta^{ab}\sigma_a M \sigma_b}$$

Hence

$$\underbrace{\underbrace{M + \delta^{ab}\sigma_a M \sigma_b}_{\sigma_0 M \tilde{\sigma}^0 + \sigma_a M \tilde{\sigma}^a}}_{\sigma_\alpha M \tilde{\sigma}^\alpha} = 2\,\mathrm{Trace}(M)\,E_2 \tag{4.1.5}$$

2)

$$\sigma_{(\alpha}\tilde{\sigma}_{\beta)} = \frac{1}{2}\left(\sigma_\alpha\tilde{\sigma}_\beta + \sigma_\beta\tilde{\sigma}_\alpha\right) \tag{4.1.6}$$

a) $\alpha = 0,\ \beta = 0 \rightsquigarrow \frac{1}{2}(2\sigma_0\tilde{\sigma}_0) = E_2$ (4.1.7)

b) $\alpha = 0,\ \beta = b \rightsquigarrow \frac{1}{2}(\sigma_0\tilde{\sigma}_b + \sigma_b\tilde{\sigma}_0)$

$= \frac{1}{2}(\sigma^b - \sigma^b) = 0$ (4.1.8)

c) $\alpha = a,\ \beta = b \leadsto \frac{1}{2}(\sigma_a \tilde{\sigma}_b + \sigma_b \tilde{\sigma}_a)$

$$= -\frac{1}{2}(\sigma^a \sigma^b + \sigma^b \sigma^a)$$

$$= -\delta^{ab} E_2 \qquad (4.1.9)$$

(Recall: $\sigma^a \sigma^b = \delta^{ab} E_2 + i \varepsilon^{ab}{}_c \sigma^c$)

Hence:

$$\sigma_{(\alpha} \tilde{\sigma}_{\beta)} = \eta_{\alpha\beta} E_2 . \qquad (4.1.10)$$

3) For any 4-tuple $X^\alpha \in \mathbb{R}^4$ have $X^\alpha \sigma_\alpha \in$ Hermitean (2×2)-matrices. If $X^\alpha = (X^0, X^1, X^2, X^3) = (X^0, \vec{X})$ and $\sigma_\alpha = (E_2, -\vec{\sigma})$ then

$$X^\alpha \sigma_\alpha = X^0 \sigma_0 - \vec{X} \cdot \vec{\sigma}$$

$$= \begin{pmatrix} X^0 - X^3 , & -X^1 + iX^2 \\ -X^1 - iX^2 , & X^0 + X^3 \end{pmatrix} \qquad (4.1.11)$$

Note

$$\det(X^\alpha \sigma_\alpha) = (X^0)^2 - \vec{X}^2$$

$$= \eta_{\alpha\beta} X^\alpha X^\beta . \qquad (4.1.12)$$

The map $\sigma : \mathbb{R}^4 \to \mathrm{Herm}(2\times 2)$,

$$X^\alpha \mapsto X^\alpha \sigma_\alpha \qquad (4.1.13)$$

is a bijection (since clearly injective and linear map between real vector-spaces of the same dimension).

So if $h \in$ Herm(2×2) then the inverse map to σ is

$$\sigma^{-1}: \text{Herm}(2\times 2) \to \mathbb{R}^4$$
$$h \mapsto X^\alpha = \tfrac{1}{2}\text{Trace}(h\tilde{\sigma}^\alpha). \qquad (4.1.14)$$

This is due to (4.1.10): if $h = X^\beta \sigma_\beta$, then

$$\text{Trace}(h\tilde{\sigma}^\alpha) = \eta^{\alpha\gamma}\,\text{Trace}(h\tilde{\sigma}_\gamma)$$
$$= \eta^{\alpha\gamma}\,\text{Trace}(X^\beta \sigma_\beta \tilde{\sigma}_\gamma)$$
$$= \eta^{\alpha\gamma} X^\beta\, \text{Trace}(\sigma_\beta \tilde{\sigma}_\gamma)$$
$$= \eta^{\alpha\gamma} X^\beta\, \text{Trace}(\sigma_{(\beta}\tilde{\sigma}_{\gamma)})$$
$$= \eta^{\alpha\gamma}\eta_{\beta\gamma} X^\beta\, \text{Trace}(E_2)$$

$\hookrightarrow$ (4.1.10)

$$= 2X^\alpha. \qquad (4.1.15)$$

$$\rightsquigarrow \quad X^\alpha = \tfrac{1}{2}\text{Trace}(h\tilde{\sigma}^\alpha).$$
$$= \tfrac{1}{2}\text{Trace}(\tilde{\sigma}^\alpha h) \qquad (4.1.16)$$

As $A(X^\alpha\sigma_\alpha)A^\dagger$ is again hermitian, it must be of the form $X'^\alpha\sigma_\alpha$, where X'^α depends linearly on X; hence

$$X'^\alpha = L^\alpha{}_\beta X^\beta \qquad (4.1.17)$$

or $$A(X^\alpha\sigma_\alpha)A^\dagger = L^\alpha{}_\beta X^\beta \sigma_\alpha \qquad (4.1.18)$$

(As (4.1.18) holds identically for all $X^\alpha \in \mathbb{R}^4$, we get

$$A \sigma_\beta A^+ = L^\alpha{}_\beta \sigma_\alpha \tag{4.1.19}$$

Using (4.1.16) we get

$$L^\alpha{}_\beta = \tfrac{1}{2} \mathrm{Trace}(\tilde{\sigma}^\alpha A \sigma_\beta A^+) \tag{4.1.20}$$

4).

$$\det\left(A (X^\alpha \sigma_\alpha) A^+\right)$$

$$= \underbrace{\det(A)}_{=1} \cdot \underbrace{\overline{\det(A)}}_{=1} \ \underbrace{\det(X^\alpha \sigma_\alpha)}_{=\eta(X,X)}$$

$$\underset{(4.1.18)}{=} \det((Lx)^\alpha \sigma_\alpha) = \eta(Lx, Lx)$$

$$\Rightarrow \quad \eta(Lx, Lx) = \eta(x,x) \tag{4.1.21}$$

$$\Leftrightarrow \quad L \in O(\mathbb{R}^4, \eta). \tag{4.1.22}$$

Moreover, from (4.1.20) get

$$L^0{}_0 = \tfrac{1}{2} \mathrm{Trace}(\underbrace{\tilde{\sigma}^0}_{=E_2} A \underbrace{\sigma_0}_{=E_2} A^+)$$

$$= \tfrac{1}{2} \mathrm{Trace}(A A^+) > 0 \tag{4.1.23}$$

$\Rightarrow$ L is time-orientation preserving.

Similarly, again from (4.1.20) get

$$L^\alpha{}_\alpha = \mathrm{Trace}(L)$$

$$= \tfrac{1}{2}\,\mathrm{Trace}(\tilde{\sigma}^\alpha A \sigma_\alpha A^+)$$

$$= \tfrac{1}{2}\,\mathrm{Trace}(A \sigma_\alpha A^+ \tilde{\sigma}^\alpha)$$

$$= \mathrm{Trace}(A\,\mathrm{Trace}(A^+)\,E_2)$$

↳ using (4.1.5)

$$= \mathrm{Trace}(A)\;\mathrm{Trace}(A^+)$$

$$= |\mathrm{Trace}(A)|^2 \geq 0 \qquad (4.1.24)$$

A space-reflection ("parity-transf.") would be given by

$$\{L^\alpha{}_\beta\} = \begin{pmatrix} 1 & \vec{0}^T \\ \vec{0} & -E_3 \end{pmatrix} \qquad (4.1.25)$$

with $\mathrm{Trace}(L) = -2$. Hence $L = p(A)$ cannot be a space-reflection.

$\Rightarrow$ $p : SL(2,\mathbb{C}) \to O(\mathbb{R}^4, \eta)$ contains neither time- nor space-reflections

$$\Rightarrow \mathrm{Image}(p) \subseteq \mathrm{Lor}^\uparrow_+ \qquad (4.1.26)$$

5) Multiplication of (4.1.19) with $\tilde{\sigma}^\beta$ gives, using (4.1.5):

$$\sigma_\alpha L^\alpha{}_\beta \tilde{\sigma}^\beta = A \sigma_\beta A^+ \tilde{\sigma}^\beta$$

$$= 2 A \operatorname{Trace}(A^+) . \qquad (4.1.27)$$

Taking the determinant of both sides gives, due to $\det(A) = 1$,

$$4 [\operatorname{Trace}(A^+)]^2 = \det(\sigma_\alpha L^\alpha{}_\beta \tilde{\sigma}^\beta)$$

or $$2 \operatorname{Trace}(A^+) = \pm \sqrt{\det(\sigma_\alpha L^\alpha{}_\beta \tilde{\sigma}^\beta)} \qquad (4.1.28)$$

Re-inserting this into (4.1.27) gives

$$A = \pm \frac{\sigma_\alpha L^\alpha{}_\beta \tilde{\sigma}^\beta}{\sqrt{\det(\sigma_\alpha L^\alpha{}_\beta \tilde{\sigma}^\beta)}} \qquad (4.1.28)$$

Finally we have that the map

$$\rho : SL(2, \mathbb{C}) \to Lor^\uparrow_+$$

$$A \mapsto \rho(A) = \{L^\alpha{}_\beta\} \qquad (4.1.29)$$

is a group homomorphism (as we should have noted already in part 4); this follows from the following: Let $A, B \in SL(2, \mathbb{C})$ with $\rho(A) = L$ and $\rho(B) = M$; that is

$$A \sigma_\beta A^+ = L^\alpha{}_\beta \sigma_\alpha \tag{4.1.30}$$

$$B \sigma_\beta B^+ = M^\gamma{}_\beta \sigma_\gamma \tag{4.1.31}$$

Then

$$\begin{aligned} & AB \sigma_\beta (AB)^+ \\ &= A B \sigma_\beta B^+ A^+ \\ &= A M^\gamma{}_\beta \sigma_\gamma A^+ \\ &= M^\gamma{}_\beta A \sigma_\gamma A^+ \\ &= M^\gamma{}_\beta L^\alpha{}_\gamma \sigma_\alpha \\ &= (L \cdot M)^\alpha{}_\beta \sigma_\alpha \end{aligned} \tag{4.1.32}$$

Hence $p(AB) = p(A) \cdot p(B)$ (4.1.33)

Also $p(E_2) = E_4$. (4.1.34)

$\Rightarrow$ p is group homomorphism.

From (4.1.28) it follows that for any $\{L^\alpha{}_\beta\}$ in a neighbourhood of E_4 there is an A s.t. $p(A) = L$. Hence p maps onto an open neighbourhood of the identity in $Lor^\uparrow_+$. Being a group homomorphism, it follows that p is surjective (since any element L of $Lor^\uparrow_+$ is the finite product of elements from the open neighbourhood (we have seen that this is true for any connected topological group).

The kernel of $p: SL(2,\mathbb{C}) \to L_{+}^{\uparrow}$ is given by those A for which

$$A \sigma_\alpha A^+ = \sigma_\alpha \qquad \forall \alpha \in \{0,1,2,3\} \tag{4.1.35}$$

For $\alpha = 0$ this implies $A A^+ = E_2$ $\Leftrightarrow$ $A^+ = A^{-1}$; hence

$$A \sigma_\alpha = \sigma_\alpha A \tag{4.1.36}$$

$\Leftrightarrow$ A commutes with <u>all</u> complex 2×2-matrices (since $\mathrm{Span}_{\mathbb{C}}\{\sigma^\alpha\} = \mathrm{End}(\mathbb{C}^2)$) $\Leftrightarrow$ $A = a E_2$, $a \in \mathbb{C}$, with $a = \pm 1$ since $\det(A) = 1$.
Hence

$$\mathrm{kernel}(p) = \{\pm E_2\}. \tag{4.1.37}$$

Problem 2

$$[P_a, P_b] = 0 \qquad (4.2.1)$$

$$[M_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b \qquad (4.2.2)$$

$$[M_{ab}, M_{cd}] = \eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} \qquad (4.2.3)$$

1)

Contraction of (4.2.2) with η^{bc} gives

$$\eta^{bc}[M_{ab}, P_c] = n P_a - P_a = (n-1) P_a . \qquad (4.2.4)$$

Here $n =$ dimension V. Hence

$$P_a = \frac{1}{n-1} \underbrace{\eta^{bc}[M_{ab}, P_c]}_{\text{sum of commutators.}} \qquad (4.2.5)$$

Contraction of (4.2.3) with η^{ad} gives

$$\eta^{ad}[M_{ab}, M_{cd}] = n M_{bc} + \eta_{bc}\eta^{ad} M_{ad} - M_{bc} - M_{bc} \qquad (4.2.6)$$

where $\eta^{ad} M_{ad} = 0$ because $\eta^{ad} = \eta^{da}$ and $M_{ad} = - M_{da}$. Hence

$$M_{bc} = \frac{1}{n-2} \eta^{ad}[M_{ab}, M_{cd}] \qquad (4.2.7)$$

where the right-hand side is again a sum of Lie-brackets.

As every basis vector P_a, M_{bc}, is the sum of Lie-brackets, so is <u>any</u> element of the Lie-algebra, which is hence perfect.

2)

Let $\rho : \mathrm{Lie}(V \rtimes O(V,\eta)) \to \mathrm{End}(W)$ a representation in the one-dim. vector space W; $\dim_{\mathbb{F}}(W) = 1$. Then all $\rho(X) \in \mathrm{End}(W)$ commute, i.e.

$$\rho(X)\rho(Y) - \rho(Y)\rho(X) = 0 \qquad (4.2.8)$$

The left hand side equals $\rho([X,Y])$. Hence the ρ-image of a commutator is zero. Since the Lie-algebra is perfect

$$\rho \equiv 0 \qquad (4.2.9)$$

This proves that a one-dimensional representation of a perfect Lie-algebra is trivial (i.e. identically zero).

In contrast, the abelian subgroup of translations has many non-trivial one-dimensional representations. For example, let $k \in V^*$, then

$$\rho(v) = \rho_k(v) := k(v) \in \mathbb{F} \qquad (4.2.10)$$

is such a representation. If $\mathbb{F} = \mathbb{C}$ we may choose

$$k = i\,\eta(p, \cdot) \qquad (4.2.11)$$

so that

$$\rho(v) = i\,\eta(p, v) = i\,p \cdot v \in i\,\mathbb{R} \qquad (4.2.12)$$

corresponding to the unitary representation

$$D(v) = \exp(i\,p \cdot v) \in \mathbb{C}, \qquad (4.2.13)$$

i.e.

$$\dot{D} = \rho_k \qquad (4.2.14)$$

It is remarkable that none of these 1-dim. rep. of V extend to $V \rtimes O(V, \eta)$.

3.)

$$P_a \mapsto \bar{P}_a := P_a \qquad (4.2.15)$$

$$M_{ab} \mapsto M'_{ab} := M_{ab} - (X_a P_b - X_b P_a) \qquad (4.2.16)$$

where $(X^1, \dots, X^n) \in \mathbb{F}^n$. It immediately follows that

$$[\bar{P}_a, \bar{P}_b] = 0 \qquad (4.2.17)$$

$$[\bar{M}_{ab}, \bar{P}_c] =$$

$$[M_{ab} - (X_a P_b - X_b P_a), P_c]$$

$$= [M_{ab}, P_c]$$

$$= \eta_{bc} P_a - \eta_{ac} P_b$$

$$= \eta_{bc} \bar{P}_a - \eta_{ac} \bar{P}_b \qquad (4.2.18)$$

$$[\bar{M}_{ab}, \bar{M}_{cd}] =$$

$$[M_{ab} - (X_a P_b - X_b P_a), M_{cd} - (X_c P_d - X_d P_c)]$$

$$= [M_{ab}, M_{cd}] - [X_a P_b - X_b P_a, M_{cd}]$$

$$- [M_{ab}, X_c P_d - X_d P_c]$$

$$= \eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac}$$

$$- X_a [P_b, M_{cd}] + X_b [P_a, M_{cd}]$$

$$- X_c [M_{ab}, P_d] + X_d [M_{ab}, P_c]$$

$$= \eta_{ad} \bar{M}_{bc} + \eta_{bc} \bar{M}_{ad} - \eta_{ac} \bar{M}_{bd} - \eta_{bd} \bar{M}_{ac} \qquad (4.2.19)$$

$$\left.\begin{aligned}
&+ \underbrace{\eta_{ad} (X_b P_c - X_c P_b)}_{1} + \underbrace{\eta_{bc} (X_a P_d - X_d P_a)}_{2} \\
&- \underbrace{\eta_{ac} (X_b P_d - X_d P_b)}_{3} - \underbrace{\eta_{bd} (X_a P_c - X_c P_a)}_{4} \\
&+ X_a (\underbrace{\eta_{bd} P_c}_{4} - \underbrace{\eta_{bc} P_d}_{2}) - X_b (\underbrace{\eta_{ad} P_c}_{1} - \underbrace{\eta_{ac} P_d}_{3}) \\
&- X_c (\underbrace{\eta_{bd} P_a}_{4} - \underbrace{\eta_{ad} P_b}_{1}) + X_d (\underbrace{\eta_{bc} P_a}_{2} - \underbrace{\eta_{ac} P_b}_{3})
\end{aligned}\right\} \text{these terms cancel}$$

Problem 3

$$\overline{T}_{(a,A)} f = f \circ \phi^{-1}_{(a,A)} \quad (4.3.1)$$

where $\phi : ((a,A), V) \mapsto \phi_{(a,A)}(V)$

$$=: AV + a \quad (4.3.2)$$

is an action.

1.)
$$\overline{T}_{(a,A)} \circ \overline{T}_{(b,B)} f$$
$$= (\overline{T}_{b,B} f) \circ \phi^{-1}_{(a,A)}$$
$$= f \circ \phi^{-1}_{(b,B)} \circ \phi^{-1}_{(a,A)}$$
$$= f \circ (\phi_{(a,A)} \circ \phi_{(b,B)})^{-1}$$
$$= f \circ \phi^{-1}_{(a+Ab, AB)}$$
$$= \overline{T}_{(a+Ab, AB)} f. \quad (4.3.3)$$

And clearly also $\overline{T}_{id_V} = id|_{C^\infty(V,\mathbb{R})}$. (4.3.4)

2) Let $s \mapsto (a(s), A(s))$ be curve in G with $a(0) = 0$, $A(0) = id_V$

$$(\overline{T}_{(a(s), A(s))} f)(X)$$
$$= f(\phi^{-1}_{(a(s),A(s))} X) = f(\phi_{(a(s),A(s))}^{-1} X) \quad (4.3.5)$$

With $(a,A)^{-1} = (-A^{-1}a, A^{-1})$ get

$$(T_{(a(s),A(s))} f)(x) = f(A^{-1}(s)(x - a(s)))$$

$$\frac{d}{ds}\Big|_{s=0} (A^{-1}(s)(x - a(s)))$$

$$= -\dot{A}x - \dot{a}$$

$$\Rightarrow \frac{d}{ds}\Big|_{s=0} (T_{(a(s),A(s))} f)(x)$$

$$= \underbrace{Df(x)}_{\text{derivative of } f \text{ at } x.} (-\dot{a} - \dot{A}x) \qquad (4.3.6)$$

3) Let

$$P_a = (e_a, 0) \qquad (4.3.7)$$

$$M_{ab} = (0, e_a \otimes \theta_b - e_b \otimes \theta_a) \qquad (4.3.8)$$

basis of Lie $(V \rtimes O(V,\eta))$

With

$$(\dot{T}_{(\dot{a},\dot{A})} f)(x) = Df(x)(-\dot{a} - \dot{A}x) \qquad (4.3.9)$$

have

$$\dot{T}_{P_a} f(x) = Df(x)(-e_a)$$

$$= -D_{e_a} f(x) = -\partial_a f(x) \qquad (4.3.10)$$

$$(\dot{T}_{M_{ab}} f)(x) = Df(x)(-M_{ab}X)$$

$$= Df(x)(-e_a X_b + e_b X_a)$$

$$= X_a \partial_b f(x) - X_b \partial_a f(x). \qquad (4.3.11)$$

$$[\dot{T}_{P_a}, \dot{T}_{P_b}] = [-\partial_a, -\partial_b] = 0 \qquad (4.3.12)$$

$$[\dot{T}_{M_{ab}}, \dot{T}_{P_c}] = [X_a\partial_b - X_b\partial_a, -\partial_c]$$

$$= (\partial_c X_a)\partial_b - (\partial_c X_b)\partial_a$$

$$= \eta_{bc}\dot{T}_{P_a} - \eta_{ac}\dot{T}_{P_b} \qquad (4.3.13)$$

$$[\dot{T}_{M_{ab}}, \dot{T}_{M_{cd}}] =$$

$$[X_a\partial_b - X_b\partial_a, X_c\partial_d - X_d\partial_c]$$

$$= \underbrace{\eta_{bc} X_a \partial_d}_{2} - \underbrace{\eta_{bd} X_a \partial_c}_{4}$$

$$- \underbrace{\eta_{ac} X_b \partial_d}_{3} + \underbrace{\eta_{ad} X_b \partial_c}_{1}$$

$$- \underbrace{\eta_{ad} X_c \partial_b}_{1} + \underbrace{\eta_{bd} X_c \partial_a}_{4}$$

$$+ \underbrace{\eta_{ac} X_d \partial_b}_{3} - \underbrace{\eta_{bc} X_d \partial_a}_{2}$$

$$= \eta_{ad}\dot{T}_{M_{bc}} + \eta_{bc}\dot{T}_{M_{ad}}$$

$$- \eta_{ac}\dot{T}_{M_{bd}} - \eta_{bd}\dot{T}_{M_{ac}} \qquad (4.3.14)$$

Problem 4

We consider Lie$(O(V,\eta))$ with V = 5-dim. real vector space with non-deg. symmetric bilinear form of signature $(1,-1,-1,-1,\sigma)$, where $\sigma = -1$ (de Sitter case) or $\sigma = 1$ (anti-de Sitter case). We know that a basis of Lie$(O(V,\eta))$ is given by the $10 = \frac{1}{2} 5 (5-1)$ element set

$$\{ M_{ab} = e_a \otimes \theta_b - e_b \otimes \theta_a \mid 0 \le a < b \le 5 \}, \qquad (4.4.1)$$

with $\{ e_a \mid 0 \le a \le 4 \}$ basis of V, $\{ \theta^a \mid 0 \le a \le 4 \}$ basis of V^* dual to $\{e_a\}$, i.e. $\theta^a(e_b) = \delta^a{}_b$, $\eta_{ab} := (e_a, e_b)$, and

$$\theta_a = \eta_{ab} \theta^b. \qquad (4.4.2)$$

Then

$$[M_{ab}, M_{cd}] = \eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} \qquad (4.4.3)$$

We write the 10 elements $M_{ab} \in V \otimes V^*$ in the following form, where now $1 \le a, b, c \le 3$

		#	
$D_a := \frac{1}{2}\,\varepsilon_a{}^{bc} M_{bc}$		3	(4.4.4)
$K_a := M_{0a}$		3	(4.4.5)
$T_a := M_{a4}$		3	
$T_0 := M_{04}$		1	
		10	

The Lie-products between them can be conveniently organised in lexicographical order

				#
$[D_a, D_b]$,	$[D_a, K_b]$,	$[D_a, T_b]$,	$[D_a, T_0]$	
(3)	(9)	(9)	(3)	(24)
	$[K_a, K_b]$,	$[K_a, T_b]$,	$[K_a, T_0]$	
	(3)	(9)	(3)	(15)
		$[T_a, T_b]$,	$[T_a, T_0]$	
		(3)	(3)	6

where the numbers in round brackets denote the numbers of independent relations. In total there are 45, which is obvious from the number 10 indep. M_{ab} and $\frac{1}{2}10(10-1) = 5 \cdot 9 = 45$ independent $[M_{ab}, M_{cd}]$.

The relations may be calculated using (4.4.3). To understand the structure, let us restrict to representative examples:

$$[D_1, D_2] = [M_{23}, M_{31}]$$

$$= \cancel{\eta_{21}} M_{33} + \eta_{33} M_{21} - \cancel{\eta_{23}} M_{31} - \cancel{\eta_{31}} M_{23}$$

$$= -M_{21} = M_{12} = D_3 \qquad (4.4.6)$$

$$[D_1, K_2] = [M_{23}, M_{02}]$$

$$= \eta_{22} M_{30} + \cancel{\eta_{30}} M_{22} - \cancel{\eta_{20}} M_{32} - \cancel{\eta_{32}} M_{20}$$

$$= -M_{30} = M_{03} = K_3 \qquad (4.4.7)$$

$$[D_1, T_2] = [M_{23}, M_{24}]$$

$$= \cancel{\eta_{24}} M_{32} + \cancel{\eta_{32}} M_{24} - \eta_{22} M_{34} - \cancel{\eta_{34}} M_{22}$$

$$= M_{34} = T_3 \qquad (4.4.8)$$

$$[D_1, T_0] = [M_{23}, M_{04}]$$

$$= \cancel{\eta_{24}} M_{30} + \cancel{\eta_{30}} M_{24} - \cancel{\eta_{20}} M_{34} - \cancel{\eta_{34}} M_{20}$$

$$= 0 \qquad (4.4.9)$$

$$[K_1, K_2] = [M_{01}, M_{02}]$$

$$= \cancel{\eta_{02}} M_{10} + \cancel{\eta_{10}} M_{02} - \eta_{00} M_{12} - \cancel{\eta_{12}} M_{00}$$

$$= -M_{12} = -D_3 \qquad (4.4.10)$$

$$[K_1, T_2] = [M_{01}, M_{24}]$$

$$= \cancel{\eta_{04}} M_{12} + \cancel{\eta_{12}} M_{04} - \cancel{\eta_{02}} M_{14} - \cancel{\eta_{14}} M_{02}$$

$$= 0 \qquad (4.4.11)$$

$$[K_1, T_1] = [M_{01}, M_{14}]$$

$$= \cancel{\eta_{04}} M_{11} + \eta_{11} M_{04} - \cancel{\eta_{01}} M_{14} - \cancel{\eta_{14}} M_{01}$$

$$= - M_{04} = - T_0 \qquad (4.4.12)$$

$$[K_1, T_0] = [M_{01}, M_{04}]$$

$$= \cancel{\eta_{04}} M_{10} + \cancel{\eta_{10}} M_{04} - \eta_{00} M_{14} - \cancel{\eta_{14}} M_{00}$$

$$= - M_{14} = - T_1 \qquad (4.4.13)$$

$$[T_1, T_2] = [M_{14}, M_{24}]$$

$$= \eta_{14} M_{42} + \eta_{42} M_{14} - \eta_{12} M_{44} - \eta_{44} M_{12}$$

$$= - \varepsilon M_{12} = - \sigma D_3 \qquad (4.4.14)$$

$$[T_1, T_0] = [M_{14}, M_{04}]$$

$$= \cancel{\eta_{14}} M_{40} + \cancel{\eta_{40}} M_{14} - \cancel{\eta_{10}} M_{44} - \eta_{44} M_{10}$$

$$= - \sigma M_{10} = \sigma M_{01} = \sigma K_1 \qquad (4.4.15)$$

In total we get

$$[D_a, D_b] = \varepsilon_{ab}{}^c D_c \tag{4.4.16}$$

$$[D_a, K_b] = \varepsilon_{ab}{}^c K_c \tag{4.4.17}$$

$$[D_a, T_b] = \varepsilon_{ab}{}^c T_c \tag{4.4.18}$$

$$[D_a, T_0] = 0 \tag{4.4.19}$$

$$[K_a, K_b] = -\varepsilon_{ab}{}^c D_c \tag{4.4.20}$$

$$[K_a, T_b] = -\delta_{ab} T_0 \tag{4.4.21}$$

$$[K_a, T_0] = -T_a \tag{4.4.22}$$

$$[T_a, T_b] = -\sigma\, \varepsilon_{ab}{}^c D_c \tag{4.4.23}$$

$$[T_a, T_0] = \sigma\, K_a \tag{4.4.24}$$

2) Consider the linear maps defined by

$$\Pi : (T_0, T_a, K_a, D_a) \mapsto (T_0, -T_a, -K_a, D_a) \tag{4.4.25}$$

$$\Theta : (T_0, T_a, K_a, D_a) \mapsto (-T_0, T_a, -K_a, D_a) \tag{4.4.26}$$

$$\Gamma : (T_0, T_a, K_a, D_a) \mapsto (-T_0, -T_a, K_a, D_a) \tag{4.4.27}$$

Have

$$\Gamma = \Theta \circ \Pi = \Pi \circ \Theta \tag{4.4.28}$$

Π = space-orientation reversal

Θ = time-orientation reversal

That Π and Θ (and hence their composition Γ) are Lie-automorphisms means that, e.g.,

$$\Pi[X, Y] = [\Pi(X), \Pi(Y)]$$

It follows by direct inspection that changing the signs of T_a and K_b or T_0 and K_a simultaneously leaves (4.4.16-24) invariant, so that Π and Θ are indeed automorphisms.

3)
$$U_1 := \mathrm{Span}\{D_a, K_b \mid 1 \le a, b \le 3\}$$

$$U_2 := \mathrm{Span}\{D_a, T_b \mid 1 \le a, b \le 3\}$$

are obviously Lie-subalgebras isomorphic to $\mathrm{Lie}(SL(2,\mathbb{C})) = \mathrm{Lie}(\mathrm{Lor})$.

a) Contraction over U_1 means to set all commutators among T_a, T_0 to zero and also all terms $\sim D_a, K_a$ between elements of U_1 and $\{T_0, T_a\}$. We thus get. This amounts to setting the right-hand sides of (4.4.23-24) to zero, thereby yielding the Lie-algebra of the Poincaré group independent of σ.

(In comparison with relations of Skript, p. 3.31, set $D_a = \dot{R}_a$, $K_a = \dot{B}_a$, $T_a = \dot{P}_a$, $T_0 = -\dot{P}_0$).

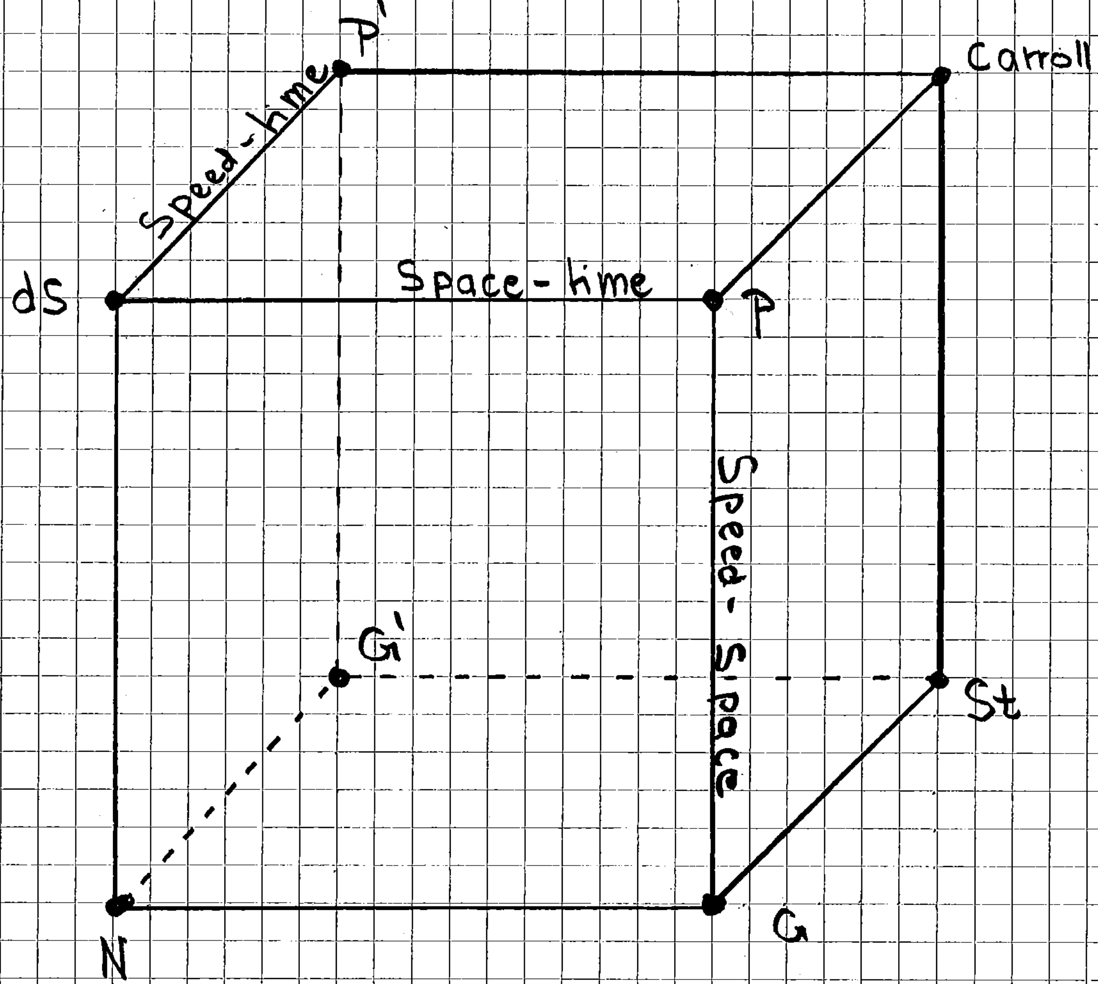
b.) contraction over $\mathcal{U}_2$ means to set all commutators among elements $\{K_a, T_0\}$ to zero and cancel all terms $\sim D_a, T_b$ in relations between $\mathcal{U}_2$ and $\{K_a, T_0\}$

$$
\begin{aligned}
[D_a, D_b] &= \varepsilon_{ab}{}^c D_c \\
[D_a, K_b] &= \varepsilon_{ab}{}^c K_c \\
[D_a, T_b] &= \varepsilon_{ab}{}^c T_c \\
[D_a, T_0] &= 0 \\
[K_a, K_b] &= 0 \\
[K_a, T_b] &= -\delta_{ab} T_0 \\
[K_a, T_0] &= 0 \\
[T_a, T_b] &= -\sigma \varepsilon_{ab}{}^c D_c \\
[T_a, T_0] &= \sigma K_a
\end{aligned}
$$

That is, in contrast to the previous case, we set the right-hand sides to (4.4.20, 22) to zero. This implies that the ensuing Lie-algebra for $\sigma = 1$ is, in fact isomorphic to that of case a), as one sees by replacing $T_a \to K_a$, $K_a \to -T_a$:

	over $\{D_a, K_b\}$ small $\{T_a, T_0\}$	over $\{D_a, T_b\}$ small $\{K_a, T_0\}$	1 & 2 2 & 1	1 & small $\{K_a, T_a\}$	2 & small $\{K_a, T_a\}$
$[D_a, D_b] = \varepsilon_{ab}{}^{c} D_c$	〃	〃	〃	〃	〃
$[D_a, K_b] = \varepsilon_{ab}{}^{c} K_c$	〃	〃	〃	〃	〃
$[D_a, T_b] = \varepsilon_{ab}{}^{c} T_c$	〃	〃	〃	〃	〃
$[D_a, T_0] = 0$	〃	〃	〃	〃	〃
$[K_a, K_b] = -\varepsilon_{ab}{}^{c} D_c$	〃	0	0	0	0
$[K_a, T_b] = -\delta_{ab} T_0$	〃	〃	〃	0	0
$[K_a, T_0] = -T_a$	〃	0	0	〃	0
$[T_a, T_b] = -\sigma \varepsilon_{ab}{}^{c} D_c$	0	〃	0	0	0
$[T_a, T_0] = \sigma K_a$	0	〃	0	0	〃
	Poincaré	para Poincaré	Carrol	Galilei	para Galilei

$\sigma = +1$; $T_a \to K_a$, $K_a \to -T_a$



The contractions indicated here are

Space-time : $\{T_a, T_0\} \longrightarrow$ small

Speed-space : $\{K_a, T_b\} \longrightarrow$ small

Speed-time : $\{K_a, T_0\} \longrightarrow$ small

These lead to P = "Poincaré",
P' = "para Poincaré", G = "Galilei",
G' = "para Galilei", C = "Carroll",
N = "non-relativistic cosmological", and
St = "static" Lie-algebras

For more discussion of these 7 contractions from dS = de Sitter algebra, see the following paper:

Henri Bacry & Jean-Marc Lévy-Leblond:
"Possible Kinematics"
Journal of Mathematical Physics,
Volume 9, Number 10, year 1968,
pages 1605 - 1614.