Sheet 5: Solutions

Problem 1

Let $\{D^a{}_b\} \in SO(3)$ in defining rep. and $\{T^{ab}\} \in \mathbb{R}^3 \otimes \mathbb{R}^3$. Then

$$(D^{(1)} \otimes D^{(1)}\, T)^{ab} = T'^{ab}$$
$$= D^a{}_c\, D^b{}_d\, T^{cd} \qquad (5.1.1)$$

Let

$$T_1^{ab} := \tfrac{1}{2}\left(T^{ab} - T^{ba}\right), \qquad (5.1.2a)$$

$$T_2^{ab} := \tfrac{1}{2}\left(T^{ab} + T^{ba}\right) - \tfrac{1}{3}\delta^{ab}\, T, \qquad (5.1.2b)$$

$$T_3^{ab} := \tfrac{1}{3}\delta^{ab}\, T, \qquad (5.1.2c)$$

where $T := \delta_{ab} T^{ab}$ (the "trace"), (5.1.2d)

be the antisymmetric, the symmetric-traceless, and the trace part; then

$$T = T_1 + T_2 + T_3 \qquad (5.1.3)$$

and

$$T_1^{ab} = -T_1^{ba} \qquad (5.1.4)$$

$$T_2^{ab} = +T_2^{ba}, \qquad (5.1.5a)$$

$$\delta_{ab} T_2^{ab} = 0 \qquad (5.1.5b)$$

Now, (anti-) symmetry and tracelessness (in general: the trace) are preserved under $D^{(1)} \otimes D^{(1)}$. This follows from the fact that (anti-) symmetrisation and taking the trace commutes with mappings by $D^{(1)} \otimes D^{(1)}$:

$$T'^{ab} = D^a{}_c D^b{}_d T^{cd}$$

$$T'^{ba} = D^b{}_c D^a{}_d T^{cd}$$

$$= D^b{}_d D^a{}_c T^{dc}$$

(by renaming indices)

$$= D^a{}_c D^b{}_d T^{dc} \qquad (5.1.6)$$

$$\delta_{ab} T'^{ab} = \underbrace{\delta_{ab} D^a{}_c D^b{}_d}_{\delta_{cd}} T^{cd}$$

$$= \delta_{cd} T^{cd}$$

$$\rightsquigarrow \quad T' = T \qquad (5.1.7)$$

Hence the linear subspaces in $\mathbb{R}^3 \otimes \mathbb{R}^3$ of antisymmetric, symmetric-traceless, and pure trace tensors are $D^{(1)} \otimes D^{(1)}$ invariant. They are of $\frac{1}{2} n(n-1)$, $\frac{1}{2} n(n+1) - 1$, and 1-dimensions for $n = 3$, i.e. 3, 5 and 1 dimension. Check: $3 + 5 + 1 = 9 = \dim(\mathbb{R}^3 \otimes \mathbb{R}^3)$. This corresponds to (Clebsch-Gordon)

$$D^{(1)} \otimes D^{(1)} = D^{(2)} \oplus D^{(1)} \oplus D^{(0)}. \qquad (5.1.8)$$

($D^{(2)}$: symm. traceless; $D^{(1)}$: antisymm; $D^{(0)}$: pure trace)

Problem 2

Consider $(D^{(1)})^{\otimes n}$ on the n-fold symmetric tensor product of $V \cong \mathbb{R}^3$. Like in Problem 1, symmetrisation or taking the trace in any pair of indices commutes with the action of $(D^{(1)})^{\otimes n}$.

If $\{T^{a_1 \cdots a_n}\} \in V \vee \cdots \vee V = V_s^{n\otimes}$, so that

$$T^{a_1 \cdots a_n} = T^{(a_1 \cdots a_n)}$$

$$:= \frac{1}{n!} \sum_{\sigma \in S_n} T^{a_{\sigma(1)} \cdots a_{\sigma(n)}} \qquad (5.2.1)$$

↑ symmetric group of n objects

The traceless elements of $V_s^{n\otimes}$ obey

$$\delta_{ab} T^{ab\, a_1 \cdots a_{n-2}} = 0 \qquad (5.2.2)$$

Note: Without loss of generality the trace is taken over the first and second index, since T is totally symmetric.

Note

$$\dim (V_s^{n\otimes}) = \frac{(n+3-1)!}{n!\,(3-1)!}$$

$$= \frac{1}{2}(n+2)(n+1) \qquad (5.2.3)$$

The number of linearly independent trace conditions (5.2.2) equals the dimension of the space of totally symmetric tensors of rank $n-2$, i.e

$$\dim\left(V_s^{(n-2)\otimes}\right) = \frac{1}{2}\, n\,(n-1). \qquad (5.2.4)$$

Hence

$$\dim\left(V_{(s,t)}^{n\otimes}\right) = \frac{1}{2}\left[(n+2)(n+1) - n(n-1)\right]$$

$$= \frac{1}{2}\left[4n+2\right] = 2n+1 \qquad (5.2.5)$$

The representation of $\vec{e}_3 \in \mathrm{Lie}(SO(3))$ under $\dot{D}^{(1)} \in \mathrm{End}(V)$

$$\dot{D}^{(1)}(\vec{e}_3)(\vec{x}) = \vec{e}_3 \times \vec{x} \qquad (5.2.6)$$

and of $J_3 = i\vec{e}_3$ under $\dot{D}^{(1)}$ on $\mathbb{C}\otimes V \cong \mathbb{C}^3$

$$\dot{D}^{(1)}(J_3)(\vec{x}) = i\vec{e}_3 \times \vec{x} \qquad (5.2.7)$$

The eigenvalues are ± 1 with eigenvectors $(\pm e_1 + i e_2) =: \vec{x}_\pm$ (5.2.8)

$$i\vec{e}_3 \times (\vec{e}_1 \pm i\vec{e}_2)$$

$$= i\vec{e}_2 \pm e_1 = \pm(e_1 \pm i\vec{e}_2) \qquad (5.2.9)$$

The representation of J_3 on $V_{(s,t)}^{n\otimes}$ is

$$J_3 \otimes \mathrm{id}_V \otimes \cdots \otimes \mathrm{id}_V + \cdots + \mathrm{id}_V \otimes \cdots \otimes \mathrm{id}_V \otimes J_3$$

$$= \left(D^{(1)} \otimes D^{(1)}\right)^{\bullet}(J_3). \qquad (5.2.10)$$

Note that $\vec{X}_+ \cdot \vec{X}_+ = (\vec{e}_1 + i\vec{e}_2)^2 = 0$; hence (5.2.11)

$$\vec{X}_+^{n\otimes} = \vec{X}_+ \otimes \cdots \otimes \vec{X}_+ \in V_{(s,t)}^{n\otimes} \quad (5.2.12)$$

i.e. it is symmetric (obviously) and traceless

$$\delta_{ab} X_+^a X_+^b \cdot X_+^c \cdots X_+^d = 0 \quad (5.2.13)$$

The eigenvalue of $(D^{(1)} \otimes D^{(1)})^{\bullet}(J_3)$ on $\vec{X}_+^{n\otimes}$ is obviously n. Then, according to the general scheme outlined in the Lecture we know that $\mathbb{C} \otimes V_{(s,t)}^{n\otimes}$ must contain a linear subspace of dimension $2n+1$ on which Lie(SO(3)) acts irreducibly. But $\dim_{\mathbb{C}} (\mathbb{C} \otimes V_{(s,t)}^{n\otimes}) = 2n+1$. So that subspace is identical to $\mathbb{C} \otimes V_{(s,t)}^{n\otimes}$. The real subspace $V_{(s,t)}^{n\otimes}$ on which Lie(SO(3)) also acts is then a $(2n+1)$-dimensional (real) irreducible space.

Problem 3

$$D(\vec{n},\alpha)\vec{x} = \vec{x} + (1-\cos\alpha)\,\vec{x}_\perp + \sin(\alpha)\,\vec{n}\times\vec{x} \qquad (5.3.1)$$

If $\vec{n}=\vec{n}(s)$, $\alpha=\alpha(s)$ with $\vec{n}(0)=\vec{n}$, $\alpha(0)=0$, $\dot{\alpha}(0)=1$ we get

$$\frac{d}{ds}\Big|_{s=0} D(\vec{n}(s),\alpha(s))\,\vec{x} = \vec{n}\times\vec{x} \qquad (5.3.2)$$

1.) Representation on $C^\infty(S^2,\mathbb{C})$ (which is an ∞-dim. complex vector space),

$$T(\vec{n},\alpha)\,f = f\circ D(\vec{n},-\alpha)$$

$$\frac{d}{ds}\Big|_{s=0} T(\vec{n},\alpha)\,f(\vec{x})$$

$$= \frac{d}{ds} f(D(\vec{n},-\alpha)\,\vec{x}) = -(\vec{n}\times\vec{x})\cdot\vec{\nabla} f(\vec{x}) \qquad (5.3.3)$$

for $\vec{x}\in S^2_1\subset\mathbb{R}^3$, i.e. $\|\vec{x}\|=1$.

Hence

$$\dot{T}_{\vec{n}}\, f(x) = -(\vec{n}\times\vec{x})\cdot\vec{\nabla} f(\vec{x})$$
$$= -\vec{n}\cdot(\vec{x}\times\vec{\nabla})\, f(\vec{x})$$
$$= -\varepsilon_{abc}\, n^a x^b \partial^c f(\vec{x}) \qquad (5.3.4)$$

$$[\dot{T}_{\vec n}, \dot{T}_{\vec m}] = [(\vec n \times \vec x)\cdot\vec\nabla, (\vec m \times \vec x)\cdot\vec\nabla]$$

$$= (\vec m \times (\vec n \times \vec x) - \vec n \times (\vec m \times \vec x))\cdot\vec\nabla$$

$$= [\vec n(\vec m\cdot\vec x) - \vec x(\vec n\cdot\vec m) - \vec m(\vec n\cdot\vec x) + \vec x(\vec n\cdot\vec m)]\cdot\vec\nabla$$

$$= -[(\vec n\times\vec m)\times\vec x]\cdot\vec\nabla$$

$$= \dot{T}_{\vec n\times\vec m}\,. \qquad (5.3.5)$$

2.) According to the general scheme we form

$$J_a := i\,\dot{T}_{\vec e_a} \qquad (5.3.6)$$

i.e.

$$J_a := -i\,(\vec e_a \times \vec x)\cdot\vec\nabla$$

$$= -i\,\varepsilon_{ab}{}^{c}\,x^b\,\partial_c \qquad (5.3.7)$$

$$\Rightarrow \quad J_1 = -i\,(y\,\partial_z - z\,\partial_y) \qquad (5.3.8a)$$

$$J_2 = -i\,(z\,\partial_x - x\,\partial_z) \qquad (5.3.8b)$$

$$J_3 = -i\,(x\,\partial_y - y\,\partial_x) \qquad (5.3.8c)$$

These vector-fields are tangential to the level surfaces of r, since

$$\dot{T}_{\vec n}\, r = -(\vec n\times\vec x)\cdot\vec\nabla r = -(\vec n\times\vec x)\cdot\frac{\vec x}{r} = 0 \qquad (5.3.9)$$

Hence $\dot{T}_{\vec{n}}$ can be expressed in terms of the vector fields $\partial/\partial\theta$ and $\partial/\partial\varphi$ in polar angles θ and φ:

$$x = r \sin\theta \cos\varphi$$
$$y = r \sin\theta \sin\varphi \qquad (5.3.10)$$
$$z = r \cos\theta$$

$$r = (x^2 + y^2 + z^2)^{1/2}$$
$$\theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \qquad (5.3.11)$$
$$\varphi = \tan^{-1}(y/x)$$

We have (using $(\tan^{-1})'(x) = (1+x^2)^{-1}$)

$$\partial_x r = \frac{x}{r} = \sin\theta\cos\varphi \qquad (5.3.12a)$$

$$\partial_y r = \frac{y}{r} = \sin\theta\sin\varphi \qquad (5.3.12b)$$

$$\partial_z r = \frac{z}{r} = \cos\theta \qquad (5.3.12c)$$

$$\partial_x \theta = (1/r)\cos\theta\cos\varphi \qquad (5.3.13a)$$

$$\partial_y \theta = (1/r)\cos\theta\sin\varphi \qquad (5.3.13b)$$

$$\partial_z \theta = (-1/r)\sin\theta \qquad (5.3.13c)$$

$$\partial_x \varphi = (-1/r)\,\frac{\sin\varphi}{\sin\theta} \tag{5.3.14a}$$

$$\partial_y \varphi = (1/r)\,\frac{\cos\varphi}{\sin\theta} \tag{5.3.14b}$$

$$\partial_z \varphi = 0 \tag{5.3.14c}$$

Hence:

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial\theta}{\partial x}\partial_\theta + \frac{\partial\varphi}{\partial x}\partial_\varphi$$

$$= \sin\theta\cos\varphi\,\partial_r + \frac{\cos\theta\cos\varphi}{r}\partial_\theta - \frac{\sin\varphi}{r\sin\theta}\partial_\varphi \tag{5.3.15a}$$

$$\partial_y = \frac{\partial r}{\partial y}\partial_r + \frac{\partial\theta}{\partial y}\partial_\theta + \frac{\partial\varphi}{\partial y}\partial_\varphi$$

$$= \sin\theta\sin\varphi\,\partial_r + \frac{\cos\theta\sin\varphi}{r}\partial_\theta + \frac{\cos\varphi}{r\sin\theta}\partial_\varphi \tag{5.3.15b}$$

$$\partial_z = \frac{\partial r}{\partial z}\partial_r + \frac{\partial\theta}{\partial z}\partial_\theta + \frac{\partial\varphi}{\partial z}\partial_\varphi$$

$$= \cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta \tag{5.3.15c}$$

The vector fields on the right-hand side of (5.3.8) are then given by

$$y\,\partial_z - z\,\partial_y$$

$$= r\sin\theta\sin\varphi\left(\cos\theta\,\partial_r - \frac{\sin\theta}{r}\,\partial_\theta\right)$$

$$- r\cos\theta\left(\sin\theta\sin\varphi\,\partial_r + \frac{\cos\theta\sin\varphi}{r}\,\partial_\theta + \frac{\cos\varphi}{r\sin\theta}\,\partial_\varphi\right)$$

$$= -\sin\varphi\,\partial_\theta - \cot\theta\cos\varphi\,\partial_\varphi \qquad (5.3.16a)$$

$$z\,\partial_x - x\,\partial_z$$

$$= r\cos\theta\left(\sin\theta\cos\varphi\,\partial_r + \frac{\cos\theta\cos\varphi}{r}\,\partial_\theta - \frac{\sin\varphi}{r\sin\theta}\,\partial_\varphi\right) - r\sin\theta\cos\varphi\left(\cos\theta\,\partial_r - \frac{\sin\theta}{r}\,\partial_\theta\right)$$

$$= \cos\varphi\,\partial_\theta - \cot\theta\sin\varphi\,\partial_\varphi \qquad (5.3.16b)$$

$$x\,\partial_y - y\,\partial_x$$

$$= r\sin\theta\cos\varphi\left(\sin\theta\sin\varphi\,\partial_r + \frac{\cos\theta\sin\varphi}{r}\,\partial_\theta + \frac{\cos\varphi}{r\sin\theta}\,\partial_\varphi\right) - r\sin\theta\sin\varphi\left(\sin\theta\cos\varphi\,\partial_r + \frac{\cos\theta\cos\varphi}{r}\,\partial_\theta - \frac{\sin\varphi}{r\sin\theta}\,\partial_\varphi\right) = \partial_\varphi \qquad (5.3.16c)$$

Hence

$$J_1 = i\,(\sin\varphi\,\partial_\theta + \cot\theta\cos\varphi\,\partial_\varphi) \qquad (5.3.17a)$$

$$J_2 = i\,(-\cos\varphi\,\partial_\theta + \cot\theta\sin\varphi\,\partial_\varphi) \qquad (5.3.17b)$$

$$J_3 = -i\,\partial_\varphi \qquad (5.3.17c)$$

and

$$\begin{aligned} J_\pm &= J_1 \pm i J_2 \\ &= i\,[(\sin\varphi \mp i\cos\varphi)\partial_\theta \\ &\qquad + \cot\theta\,(\cos\varphi \pm i\sin\varphi)\,\partial_\varphi] \\ &= \pm\, e^{\pm i\varphi}\,\partial_\theta + i\cot\theta\, e^{\pm i\varphi}\,\partial_\varphi \\ &= e^{\pm i\varphi}\,(\pm\partial_\theta + i\cot\theta\,\partial_\varphi) \end{aligned} \qquad (5.3.18)$$

In order to calculate $\vec{J}^2$ we may either sum $J_1^2 + J_2^2 + J_3^2$ from or use that

$$\begin{aligned} J_+ J_- &= (J_1 + iJ_2)(J_1 - iJ_2) \\ &= J_1^2 + J_2^2 - i\,[J_1, J_2] \\ &= \vec{J}^2 - J_3^2 + J_3 \end{aligned} \qquad (5.3.19)$$

so that

$$\vec{J}^2 = J_+ J_- + J_3^2 - J_3 \qquad (5.3.20)$$

Now,

$$J_+ J_- = e^{i\varphi}(\partial_\theta + i\cot\theta\,\partial_\varphi)\,e^{-i\varphi}(-\partial_\theta + i\cot\theta\,\partial_\varphi)$$

$$= (\partial_\theta + i\cot\theta\,\partial_\varphi + \cot\theta)(-\partial_\theta + i\cot\theta\,\partial_\varphi)$$

$$= (-\partial_\theta^2 - i\frac{1}{\sin^2\theta}\partial_\varphi + i\cot\theta\,\partial^2_{\theta\varphi}$$

$$+ i\cot\theta\,\partial^2_{\varphi\theta} - \cot^2\theta\,\partial^2_\varphi - \cot\theta\,\partial_\theta$$

$$+ i\cot^2\theta\,\partial_\varphi)$$

$$= (-\partial_\theta^2 - i\partial_\varphi - \cot^2\theta\,\partial^2_\varphi - \cot\theta\,\partial_\theta) \qquad (5.3.21)$$

hence

$$\vec{J}^2 = J_+ J_- - \partial_\varphi^2 + i\partial_\varphi$$

$$= (-\partial_\theta^2 - \frac{1}{\sin^2\theta}\partial_\varphi^2 - \cot\theta\,\partial_\theta)$$

$$= -\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) - \frac{1}{\sin^2\theta}\partial^2_\varphi \qquad (5.3.22)$$

Using the differential operators J_3 and $J_\pm$ we determine $\varphi_\lambda : S^2 \to \mathbb{C}$ so that

$$J_3 \psi_\ell = \ell \psi_\ell \qquad (5.3.23a)$$

$$J_+ \psi_\ell = 0 \qquad (5.3.23b)$$

The first leads to

$$\psi_\ell(\theta,\varphi) = f_\ell(\theta)\, e^{i\ell\varphi} \qquad (5.3.24)$$

The second is then equivalent to

$$(\partial_\theta + i \cot\theta\, \partial_\varphi)\, f_\ell(\theta)\, e^{i\ell\varphi} = 0$$

$$\Leftrightarrow \quad f'_\ell(\theta) = \ell \cot\theta\, f_\ell(\theta)$$

$$\Leftrightarrow \quad f'_\ell / f_\ell = \ell \cot\theta = \ell \frac{(\sin\theta)'}{\sin\theta}$$

$$\Leftrightarrow \quad f_\ell(\theta) = C_\ell (\sin\theta)^\ell$$

$$\Rightarrow \quad \psi_\ell(\theta,\varphi) = C_\ell (\sin\theta)^\ell e^{i\ell\varphi} \qquad (5.3.25)$$

The constants C_ℓ are determined by the requirement that the ψ_ℓ be normalised

$$1 = \langle \psi_\ell | \psi_\ell \rangle = \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin\theta\, |\psi_\ell(\theta,\varphi)|^2$$

$$= 2\pi C_\ell^2 \int_0^\pi d\theta\, (\sin\theta)^{2\ell+1} \qquad (5.3.26)$$

Now,

$$I_{2\ell+1} := \int_0^{\pi} d\theta \, (\sin\theta)^{2\ell+1}$$

$$= \int_0^{\pi} d\theta \, (\sin\theta)^{2\ell} (-\cos\theta)'$$

$$= -\cos\theta \, (\sin\theta)^{2\ell} \Big|_0^{\pi} + \int d\theta \cos\theta \left[(\sin\theta)^{2\ell} \right]'$$

$$= 2\ell \int_0^{\pi} d\theta \, (\sin\theta)^{2\ell-1} \cos^2\theta$$

$$= 2\ell \, I_{2\ell-1} - 2\ell \, I_{2\ell+1} \qquad (5.3.27)$$

$$\Rightarrow \quad I_{2\ell+1} = \frac{2\ell}{2\ell+1} \, I_{2\ell-1} \qquad (5.3.28)$$

This recursion relation is solved by

$$I_{2\ell+1} = \frac{(2\ell)!!}{(2\ell+1)!!} \, I_1 = 2 \cdot \frac{(2\ell)!!}{(2\ell+1)!!} \qquad (5.3.29)$$

since

$$I_1 = \int_0^{\pi} d\theta \, \sin\theta = 2 \qquad (5.3.30)$$

Note

$$(2\ell)!! = 2 \cdot 4 \cdot 6 \cdots 2\ell \quad \text{(even prod.)} \qquad (5.3.31a)$$

$$(2\ell+1)!! = 1 \cdot 3 \cdot 5 \cdots (2\ell+1) \quad \text{(odd prod.)} \qquad (5.3.31b)$$

The normalisation condition gives

$$1 = 4\pi \, C_l^2 \, \frac{(2l)!!}{(2l+1)!!}$$

$$\curvearrowright \quad C_l = \frac{1}{\sqrt{4\pi}} \sqrt{\frac{(2l+1)!!}{(2l)!!}} \qquad (5.3.32)$$

$$\curvearrowright \quad \psi_l(\theta,\varphi) = \sqrt{\frac{(2l+1)!!}{4\pi(2l)!!}} \, (\sin\theta)^l \, e^{il\varphi} \qquad (5.3.33)$$

The other ψ_m, $-l \leq m \leq +l$, are now obtained by successive application of J_- and normalisation.

Recall

$$J_- = e^{-i\varphi}\left(-\partial_\theta + i \cot\theta \, \partial_\varphi\right) \qquad (5.3.34)$$

and

$$\partial_\varphi \psi_l = i l \psi_l , \qquad (5.3.35a)$$

$$\partial_\theta \psi = l \cot\theta \, \psi_l . \qquad (5.3.35b)$$

Hence

$$J_- \psi_l = e^{-i\varphi}\left(-l\cot\theta - l\cot\theta\right)\psi_l$$

$$= -2l\cot\theta \, e^{-i\varphi} \psi_l \qquad (5.3.36)$$

$$\psi_{l-1} = \frac{J_- \psi_l}{\sqrt{l(l+1) - l(l-1)}} = \frac{J_- \psi_l}{\sqrt{2l}}$$

$$= -\sqrt{2l} \left[\frac{(2l+1)!!}{4\pi (2l)!!} \right]^{1/2} \cos\theta \, (\sin\theta)^{l-1} e^{i(l-1)\varphi} \tag{5.3.37}$$

Example: $l = 1$

$$\psi_1(\theta,\varphi) = \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{i\varphi} \tag{5.3.38a}$$

$$\psi_0(\theta,\varphi) = -\left(\frac{3}{4\pi}\right)^{1/2} \cos\theta \tag{5.3.38b}$$

$$J_- \psi_0(\theta,\varphi) = e^{-i\varphi}(-\partial_\theta + i \cot\theta \, \partial_\varphi)\psi_0$$
$$= -\left(\frac{3}{4\pi}\right)^{1/2} \sin\theta \, e^{-i\varphi}$$

normalised:

$$\psi_-(\theta,\varphi) = -\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{-i\varphi} \tag{5.3.38c}$$

Different areas of application employ different phase conventions. The highest J_3-eigenvalue function in $C^\infty(S^2, \mathbb{C})$, Y_{ll}, is defined in QM as follows:

$$Y_{ll}(\theta,\varphi) = (-1)^l \psi_l$$

$$= (-1)^l \left[\frac{(2l+1)!!}{4\pi (2l)!!} \right]^{1/2} (\sin\theta)^l e^{il\varphi} \tag{5.3.39}$$

and then recursively as follows

$$Y_{l(m-1)} := \frac{J_- Y_{lm}}{\hbar\sqrt{l(l+1)-m(m-1)}} \qquad (5.3.40)$$

for $l \geq m \geq -l+1$

Then we would have obtained

$$Y_{11}(\theta,\varphi) = -\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{i\varphi} \qquad (5.3.41a)$$

$$Y_{10}(\theta,\varphi) = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta \qquad (5.3.41b)$$

$$Y_{1-1}(\theta,\varphi) = \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{-i\varphi} \qquad (5.3.41c)$$

Problem 4

$\vee^n V^*$ = n-fold symmetrised tensor-product of V^*

Let $\Theta \in \vee^n V^*$; to show $\exists\ \phi^{(1)}, \dots, \phi^{(n)} \in V^*$ s.t.

$$\Theta = \phi^{(1)} \vee \dots \vee \phi^{(n)} \tag{5.4.1}$$

Lemma: Let $\Theta, \Theta' \in \vee^n V^*$ and

$$\Theta(v, \dots, v) = \Theta'(v, \dots, v) \quad \forall v \in V \tag{5.4.2}$$

↘ n-times the <u>same</u> entity ↙

then $\Theta = \Theta'$. (Note: This generalises the result, that a bilinear symmetric form is uniquely determined by its quadratic form).

Proof: Set $v = u + \lambda w$; then (5.4.3)

$$\begin{aligned}\Theta(v, \dots, v) = \ & \Theta(u, \dots, u) \\ & + \binom{n}{1} \lambda\, \Theta(w, u, \dots, u) \\ & + \binom{n}{2} \lambda^2\, \Theta(w, w, u, \dots, u) \\ & \vdots \\ & + \binom{n}{n} \lambda^n\, \Theta(w, \dots, w)\end{aligned} \tag{5.4.4}$$

and the corresponding equation for Θ'. As equality (5.4.2) holds $\forall v$ the equality of (5.4.4) with the corresponding expansion of Θ' in terms of powers of λ holds identically in λ. Hence, in particular, the terms $\sim \lambda$ must be the same:

$$\Theta(w, u, \cdots, u) = \Theta'(w, u, \cdots, u) \qquad (5.4.5)$$

$$\forall\, w, u \in V.$$

Fixing w and setting $u = x + \lambda y$ we conclude as before

$$\Theta(w, y, x, \cdots, x) = \Theta'(w, y, x, \cdots, x) \qquad (5.4.6)$$

$$\forall\, w, y, x \in V$$

Proceeding as before we finally arrive at

$$\Theta(v_1, v_2, \cdots, v_n) = \Theta(v_1, v_2, \cdots, v_n) \qquad (5.4.7)$$

$$\forall\, v_1, \cdots, v_n \in V$$

i.e. $\Theta = \Theta'$. q.e.d.

In view of this Lemma it suffices to show that there exist $\phi^{(1)}, \cdots, \phi^{(n)} \in V^*$ such that

$$(\Theta - \phi^{(1)} \vee \cdots \vee \phi^{(n)})(v, \cdots, v) = 0 \qquad (5.4.8)$$

for all $v \in V^*$

Consider v of the form $v = \binom{1}{z}$, $z \in \mathbb{C}$, then (5.4.8) is equivalent to

$$\Theta(v, \dots, v) = (\phi_0^{(1)} + \phi_1^{(1)} z) \cdots (\phi_0^{(n)} + \phi_1^{(n)} z) \qquad (5.4.9)$$

The left-hand side is a polynomial of degree n in the complex variable z. The right hand-side is its factorization into monomials. We know that n-pairs of complex numbers $(\phi_0^{(1)}, \phi_1^{(1)})$, $\dots$, $(\phi_0^{(n)}, \phi_1^{(n)})$ exist by the fundamental theorem of algebra. From that theorem we also know that the n 2-tuples $(\phi_0^{(1)}, \phi_1^{(1)}), \dots, (\phi_0^{(n)}, \phi_1^{(n)})$ are uniquely determined up to permutation. Finally note that equality in (5.4.9) for all z implies in particular equality of the coefficients of z^n on both sides, i.e.

$$\Theta_{1 \cdots 1} = \phi_1^{(1)} \cdots \phi_1^{(n)} \qquad (5.4.10)$$

showing that (5.4.8) also holds for vectors $v = \binom{0}{1}$. Hence (5.4.8) holds for $v = \binom{1}{z}$ and $v = \binom{0}{1}$ and clearly all multiples of those, and hence for <u>all</u> $v \in \mathbb{C}^2$.

Problem 5

Let V be a complex vector space. An antilinear map

$$C : V \to V \qquad (5.5.1)$$

satisfying

$$C \circ C = id_V \qquad (5.5.2)$$

is called a real structure. A $v \in V$ is called real iff $C(v) = v$.

Remark: If V is a real vector space and $V^{\mathbb{C}} := (\mathbb{C} \otimes V)^{\mathbb{C}}$ its complexification, then $V^{\mathbb{C}}$ has a natural real structure given by

$$C(z \otimes v) = \bar{z} \otimes v \qquad (5.5.3)$$

plus $\mathbb{C}$-antilinear extension to all of $V^{\mathbb{C}}$. The real vectors in $V^{\mathbb{C}}$ are then those of the form $1 \otimes v$ (note that $r \otimes v = 1 \otimes rv$ for $r \in \mathbb{R}$).

We know that there is a natural anti-isomorphism

$$j : V \to \bar{V} \qquad (5.5.4)$$

1)

Then

$$K := j \circ C : V \to \bar{V} \qquad (5.5.5)$$

is linear and a bijection, hence a (linear) isomorphism. The condition (5.5.2) is just $C^{-1} = C$, hence in terms of K and j equivalent to

$$j^{-1} \circ K = K^{-1} \circ j \qquad (5.5.6a)$$

or

$$K^{-1} = j^{-1} \circ K \circ j^{-1} \qquad (5.5.6b)$$

Conversely given a linear map $K : V \to \bar{V}$ satisfying (5.5.6) we define a real structure by

$$C := j^{-1} \circ K \qquad (5.5.7)$$

2) Let $W = V \otimes \bar{V}$ and (5.5.8)

$$Ex : V \otimes \bar{V} \to \bar{V} \otimes V \qquad (5.5.9)$$

$$v \otimes \bar{u} \mapsto Ex(v \otimes \bar{u}) = \bar{u} \otimes v$$

plus linear extension

Define

$$C := Ex \circ j \otimes j^{-1} : W \to W \qquad (5.5.10)$$

so that

$$C(v \otimes \bar{u}) = j^{-1}(\bar{u}) \otimes j(v)$$

+ antilinear extension

then

$$C \circ C(v \otimes \bar{u})$$
$$= C(j^{-1}(\bar{u}) \otimes j(v))$$
$$= v \otimes \bar{u}$$
$$\Rightarrow \quad C \circ C = id_W \qquad (5.5.11)$$

Likewise, if $W = V \oplus \bar{V}$ and (5.5.12)

$$Ex : V \oplus \bar{V} \to \bar{V} \oplus V \quad \text{(linear)} \qquad (5.5.13)$$
$$(v, \bar{u}) \to Ex(v, \bar{u}) := (\bar{u}, v)$$

Define $C := Ex \circ (j^{-1} \times j) : W \to W$ (5.5.14)

then

$$C \circ C(v, \bar{u}) = C(j^{-1}(\bar{u}), j(v))$$
$$= (v, \bar{u})$$

and C is an antilinear involution.

The real vectors in $V \otimes \bar{V}$ are

$$\text{Real}(V \otimes \bar{V}) = \text{Span}_{\mathbb{R}}\{ v \otimes j(v) : v \in V \}$$

and in $V \oplus \bar{V}$

$$\text{Real}(V \oplus \bar{V}) = \{ (v, j(v)) : v \in V \} \qquad (5.5.15)$$

3)

If $\quad W = V \oplus \bar{V}^*$ (5.5.16)

we get a real structure provided we have an inner product

$$\varepsilon : V \times V \to \mathbb{C} \tag{5.5.17}$$

i.e. a non-degenerate bilinear form (no assumptions concerning symmetry here). ε gives us an isomorphism

$$\varepsilon_\downarrow : V \to V^*, \tag{5.5.18a}$$

$$\varepsilon_\uparrow := (\varepsilon_\downarrow)^{-1} : V^* \to V. \tag{5.5.18b}$$

Together with the natural anti-isomorphism

$$j : V \to \bar{V}, \tag{5.5.19}$$

hence $\quad j^T : \bar{V}^* \to V^*,$ (5.5.20)

we get

$$\begin{array}{ccc} V & \xrightarrow{\varepsilon_\downarrow} & V^* \\ j \big\downarrow & /\!/\!/ & \big\downarrow j^* = (j^T)^{-1} \\ \bar{V} & \xrightarrow[\bar{\varepsilon}_\downarrow]{} & \bar{V}^* \end{array} \tag{5.5.21}$$

where

$$j^* := (j^T)^{-1} \qquad (5.5.22)$$

This defines $\bar{\varepsilon}_\downarrow$ and its inverse:

$$\bar{\varepsilon}_\downarrow = (j^T)^{-1} \circ \varepsilon_\downarrow \circ j^{-1}$$
$$: \bar{V} \to \bar{V}^*, \qquad (5.5.23)$$

and

$$\bar{\varepsilon}_\uparrow := (\bar{\varepsilon}_\downarrow)^{-1}$$
$$= j \circ \varepsilon_\uparrow \circ j^T$$
$$: \bar{V}^* \to \bar{V} . \qquad (5.5.24)$$

Now, in order to give a real structure to $W = V \oplus \bar{V}^*$ we can use the ε-induced isomorphicities between V and V^* and $\bar{V}$ and $\bar{V}^*$ to exchange the factors. Hence we set

$$C : W \to W$$
$$:= Ex \circ (j^* \circ \varepsilon_\downarrow \times j^{-1} \circ \bar{\varepsilon}_\uparrow)$$
$$= Ex \circ ((j^T)^{-1} \circ \varepsilon_\downarrow \times \varepsilon_\uparrow \circ j^T) \qquad (5.5.25)$$

$$C(\varphi, \bar{z}) = (\varepsilon_\uparrow(j^T(\bar{z})), (j^T)^{-1}(\varepsilon_\downarrow(\varphi)))$$

Clearly $(j^T)^{-1} \circ \varepsilon_\downarrow$ and $\varepsilon_\uparrow \circ j^T$ are antilinear and inverses of each other, so that $C \circ C = \mathrm{id}_W$. The real vectors $(\varphi, \bar{\chi})$ are then given by

$$\mathrm{Real}\,(V \oplus \bar{V}^*) = \{ (v, (j^T)^{-1}(\varepsilon\downarrow(v))) : v \in V \} \qquad (5.5.26)$$

In index notation, with respect to adapted bases, we have for

$$\chi \in V \oplus \bar{V}^*$$

$$\chi = \begin{pmatrix} \varphi^A \\ \psi_{\dot{A}} \end{pmatrix} \begin{matrix} \leftarrow \in V \\ \leftarrow \in \bar{V}^* \end{matrix} \qquad (5.5.27)$$

If e_A, θ^A are dual bases to V and V^* and

$$\bar{e}_{\dot{A}} := j(e_A) \qquad (5.5.28a)$$

$$\bar{\theta}^{\dot{A}} : j^*(\theta^A) = (j^T)^{-1}(\theta^A) \qquad (5.5.28b)$$

the corresponding dual bases in the complex-conjugate spaces, we have for $\varphi = \varphi^A e_A$, $\psi = \psi_{\dot{A}} \bar{\theta}^{\dot{A}}$

$$(j^T)^{-1}(\varepsilon\downarrow(\varphi)) = (j^T)^{-1}(\varphi^A \varepsilon\downarrow(e_A))$$

$$= \bar{\varphi}^{\dot{A}} (j^T)^{-1}(\varepsilon_{AB}\theta^B)$$

$$= \bar{\varphi}^{\dot{A}} \bar{\varepsilon}_{\dot{A}\dot{B}} \bar{\theta}^{\dot{B}} =: \bar{\varphi}_{\dot{B}} \bar{\theta}^{\dot{B}} \qquad (5.5.29)$$

where

$$\bar{\psi}_{\dot{B}} := \bar{\psi}^{\dot{A}} \bar{\varepsilon}_{\dot{A}\dot{B}} . \qquad (5.5.30)$$

Likewise

$$\varepsilon_{\uparrow}(j^T(\psi)) = \varepsilon_{\uparrow}(j^T(\psi_{\dot{A}} \bar{\theta}^{\dot{A}}))$$

$$= \varepsilon_{\uparrow}(\bar{\psi}_A \theta^A) = \bar{\psi}_A \varepsilon^{BA} e_B$$

$$=: \bar{\psi}^B e_B \qquad (5.5.31)$$

where $\bar{\psi}^B = \varepsilon^{BA} \bar{\psi}_A$ (5.5.32)

Therefore, in components

$$C \begin{pmatrix} \varphi^A \\ \psi_{\dot{A}} \end{pmatrix} = \begin{pmatrix} \varepsilon^{AB} \bar{\psi}_B \\ \bar{\varphi}^{\dot{B}} \bar{\varepsilon}_{\dot{B}\dot{A}} \end{pmatrix} \qquad (5.5.33)$$

Spinors of the form

$$\chi = \begin{pmatrix} \varphi^A \\ \psi_{\dot{A}} \end{pmatrix} \in V \oplus \bar{V}^* \qquad (5.5.34)$$

are called Dirac-spinors; those <u>real</u> ones are of the form

$$\chi = \begin{pmatrix} \varphi^A \\ \bar{\varphi}_{\dot{A}} \end{pmatrix} \qquad (5.5.35)$$

and are called Majorana spinors.

Remark: On page S5.24 we used the notion of the "transposed map" j^T, even though j is not linear but rather anti-linear. This is defined just as for the linear case:

In general, let

$$f : V \to W \tag{5.5.36}$$

be an antilinear map between complex vector spaces V and W; then

$$f^T : W^* \to V^* \tag{5.5.37}$$

is an antilinear map defined by

$$f^T(\omega)(v) := \omega(f v). \tag{5.5.38}$$

$\forall\ \omega \in W^*$ and $\forall\ v \in V$. Antilinearity of f^T follows from that of f:

$[f^T(a_1\omega_1 + a_2\omega_2)](v)$

$= a_1\omega_1(f(v)) + a_2\,\omega_2(f(v))$

$= \omega_1(a_1 f(v)) + \omega_2(a_2 f(v))$

$= \omega_1\, f(\bar{a}_1 v) + \omega_2(f(\bar{a}_2 v))$

$= f^T(\omega_1)(a_1 v) + f^T(\omega_2)(\bar{a}_2 v)$

$$= [\bar{a}_1 f^T(\omega_1) + \bar{a}_2\, f^T(\omega_2)](v). \tag{5.5.39}$$

Problem 6

Before we turn to Lie-algebras, we consider the given situation on the level of vector spaces.

- Let V be a real vector space with complex structure

$$J : V \to V \quad \in \quad End(V) \tag{5.6.1}$$

s.t. $$J \circ J = - id_V \tag{5.6.2}$$

Note that (5.6.2) implies

$$[\det(J)]^2 = (-1)^n \tag{5.6.3}$$

where $n = \dim_{\mathbb{R}}(V)$. Since the left-hand side of (5.6.3) is non-negative we must have n = even; that is: complex structure exist only on even-dimensional vector spaces. In that case

$$[\det(J)]^2 = 1 \Rightarrow \det(J) = \pm 1 \tag{5.6.4}$$

so that J is an isomorphism, i.e.

$$J \in GL(V) \tag{5.6.5}$$

Using J we may turn V into a complex vector space by defining scalar $\mathbb{C}$ multiplication on the set V

by

$$(a+ib)\cdot v := av + bJ(v) \qquad (5.6.6)$$

We denote the complex vector space so defined by $V^{\mathbb{C}}$. Note that <u>as sets</u> $V = V^{\mathbb{C}}$ and that

$$\dim_{\mathbb{R}}(V) = 2\,\dim_{\mathbb{C}}(V^{\mathbb{C}}) \qquad (5.6.7)$$

Independent of the existence of J we may consider the real vector space $(\mathbb{C}\otimes_{\mathbb{R}} V)$. This is now a <u>new set</u> and

$$\dim_{\mathbb{R}}(\mathbb{C}\otimes_{\mathbb{R}} V) = 2\,\dim_{\mathbb{R}}(V). \qquad (5.6.8)$$

$(\mathbb{C}\otimes_{\mathbb{R}} V)$ has a natural complex structure which for the sake of this exercise we denote by J_2; it is given by

$$J_2 : (\mathbb{C}\otimes_{\mathbb{R}} V) \to (\mathbb{C}\otimes_{\mathbb{R}} V)$$

$$z\otimes v \mapsto iz\otimes v \qquad (5.6.9)$$

plus $\mathbb{R}$-linear extension.

Using J_2 we may turn the real vector space $(\mathbb{C}\otimes_{\mathbb{R}} V)$ into a complex vector space $(\mathbb{C}\otimes_{\mathbb{R}} V)^{\mathbb{C}}$, called <u>the complexification of V</u>, by

$$(a+ib)\,(z \otimes_{\mathbb{R}} v)$$

$$= z \otimes_{\mathbb{R}} (av) + J_2\,(z \otimes_{\mathbb{R}} (bv))$$

$$= z \otimes_{\mathbb{R}} (av) + iz \otimes (bv)$$

$$= az \otimes_{\mathbb{R}} v + ibz \otimes_{\mathbb{R}} v$$

$$= (a+ib)z \otimes_{\mathbb{R}} v \qquad (5.6.10)$$

+ $\mathbb{R}$-linear extension on sums of terms $\sum_i z_i \otimes_{\mathbb{R}} v_i$.

- We now consider the real vector space $\mathbb{C} \otimes_{\mathbb{R}} V$ and the complex vector space $(\mathbb{C} \otimes_{\mathbb{R}} V)^{\mathbb{C}}$ for those V which are already equipped with a complex structure J, as explained above.

  Let us look at the real space $\mathbb{C} \otimes_{\mathbb{R}} V$ first. We can extend $J: V \to V$ to

$$J_1 : (\mathbb{C} \otimes_{\mathbb{R}} V) \to (\mathbb{C} \otimes_{\mathbb{R}} V) \qquad (5.6.11)$$

via

$$J_1(z \otimes_{\mathbb{R}} v) := z \otimes_{\mathbb{R}} J(v) \qquad (5.6.12)$$

+ $\mathbb{R}$-linear extension,

i.e.

$$J_1 := \mathrm{id}_{\mathbb{C}} \otimes_{\mathbb{R}} J. \qquad (5.6.13)$$

As elements of End$(\mathbb{C} \otimes_{\mathbb{R}} V)$ the maps J_1 and J_2 commute since

$$J_1 = id_{\mathbb{C}} \otimes_{\mathbb{R}} J \qquad (5.6.14a)$$

$$J_2 = i \; id_{\mathbb{C}} \otimes_{\mathbb{R}} id_V \qquad (5.6.14b)$$

affect only the second and first tensor factor respectively; hence

$$[J_1, J_2] = 0 . \qquad (5.6.15)$$

Clearly

$$J_1 \circ J_1 = J_2 \circ J_2 = - id_{\mathbb{C} \otimes V} \qquad (5.6.16)$$

We now consider

$$P_\pm \in \text{End}(\mathbb{C} \otimes_{\mathbb{R}} V) \qquad (5.6.17)$$

defined by

$$P_\pm := \frac{1}{2}\left(id_{\mathbb{C} \otimes V} \pm J_1 \circ J_2 \right) \qquad (5.6.18)$$

Note that

$$(J_1 \circ J_2)^2 = J_1 \circ J_2 \circ J_1 \, J_2$$

$$(5.6.15) \longrightarrow \quad = J_1 \circ J_1 \circ J_2 \circ J_2$$

$$(5.6.16) \longrightarrow \quad = (- id_{\mathbb{C} \otimes V}) \circ (- id_{\mathbb{C} \otimes V})$$

$$= id_{\mathbb{C} \otimes V} \qquad (5.6.19)$$

Hence

$$P_\pm \circ P_\pm = \frac{1}{4}\left(id \pm 2 J_1 \circ J_2 + (J_1 \circ J_2)^2 \right)$$

$$= P_\pm \,, \qquad (5.6.20a)$$

$$P_\pm \circ P_\mp = \frac{1}{4}(id \pm J_1 \circ J_2)(id \mp J_1 \circ J_2)$$

$$= \frac{1}{4}\left(id - (J_1 \circ J_2)^2 \right) = 0 \,, \qquad (5.6.20b)$$

and clearly

$$P_+ + P_- = id_{\mathbb{C}\otimes V} \,. \qquad (5.6.20c)$$

Equations (5.6.20) say that $P_\pm$ are projection operators with

$$\text{Image}(P_\pm) = \text{Kernel}(P_\mp) \qquad (5.6.21)$$

and

$$\mathbb{C} \otimes_{\mathbb{R}} V = \text{Image}(P_+) \oplus \text{Image}(P_-)$$
$$= \text{Kernel}(P_-) \oplus \text{Kernel}(P_+) \qquad (5.6.22)$$

Now, $V \in \text{Kernel}(P_\pm)$

$\Leftrightarrow \quad 0 = P_\pm(V) \quad \Leftrightarrow \quad V \pm J_1 \circ J_2 (V) = 0$

$\Leftrightarrow \quad J_1(V) \mp J_2(V) = 0$

$\Leftrightarrow \quad J_1(V) = \pm J_2(V) \qquad (5.6.23)$

All this means that if the real vector-space V has a complex structure $J: V \to V$, then $\mathbb{C} \otimes_{\mathbb{R}} V$ has a J-induced direct-sum decomposition

$$\mathbb{C} \otimes_{\mathbb{R}} V = (\mathbb{C} \otimes_{\mathbb{R}} V)_+ \oplus (\mathbb{C} \otimes_{\mathbb{R}} V)_- \qquad (5.6.24a)$$

where

$$(\mathbb{C} \otimes_{\mathbb{R}} V)_\pm := \{ v \in \mathbb{C} \otimes_{\mathbb{R}} V : J_1(v) = \pm J_2(v) \}. \qquad (5.6.24b)$$

If instead of the real vector space $(\mathbb{C} \otimes_{\mathbb{R}} V)$ we considered the complexified $(\mathbb{C} \otimes_{\mathbb{R}} V)^{\mathbb{C}}$, the reasoning would be along the very same lines. The only change would be that J_1 is simply written as multiplication with i and that $J_2 = id_{\mathbb{C}} \otimes_{\mathbb{R}} J$ extends to a $\mathbb{C}$-linear map because of (5.6.15), i.e. commutativity with J_1. The projection operators are now

$$P_\pm \in End((\mathbb{C} \otimes_{\mathbb{R}} V)^{\mathbb{C}}) \qquad (5.6.25a)$$

$$P_\pm := \tfrac{1}{2}(id_{(\mathbb{C} \otimes V)^{\mathbb{C}}} \pm i J_2) \qquad (5.6.25b)$$

 and the direct-sum decomposition

reads

$$(\mathbb{C}\otimes_{\mathbb{R}}V)^{\mathbb{C}} = (\mathbb{C}\otimes_{\mathbb{R}}V)^{\mathbb{C}}_{+} \oplus (\mathbb{C}\otimes_{\mathbb{R}}V)^{\mathbb{C}}_{-} \quad (5.6.26a)$$

with

$$(\mathbb{C}\otimes_{\mathbb{R}}V)^{\mathbb{C}}_{\pm} = \{v\in(\mathbb{C}\otimes_{\mathbb{R}}V)^{\mathbb{C}} : J_2(v)=\pm i v\} \quad (5.6.26b)$$

Finally we can lift the whole discussion to the level of Lie-algebras. For that we have to recall the following two facts

1) A complex structure on a real Lie-algebra L is a complex structure $C: L\to L$ of the underlying real vector space (here denoted by L instead of V) which, in addition to $C\circ C = -\mathrm{id}_L$ satisfies $\forall\, X,Y\in L$:

$$C([X,Y]) = [CX,Y] = [X,CY] \quad (5.6.27)$$

2) $\mathbb{C}\otimes_{\mathbb{R}} L$ is made into a real Lie-algebra by defining

$$[Z_1\otimes X_1, Z_2\otimes X_2] := Z_1Z_2\otimes[X_1,X_2] \quad (5.6.28)$$

+ $\mathbb{R}$-linear extension.

The map

$$J_2 \, (Z \otimes X) = iZ \otimes X \qquad (5.6.29)$$

+ $\mathbb{R}$-linear extension

is then a complex structure on $\mathbb{C} \otimes L$ since (5.6.28) implies that it satisfies (5.6.27).

Now consider the extension J_1 of J to $\mathbb{C} \otimes_{\mathbb{R}} L$

$$J_1 = id_{\mathbb{C}} \otimes_{\mathbb{R}} J \qquad (5.6.30)$$

We have

$$J_1 ([Z_1 \otimes X_1, Z_2 \otimes X_2])$$

$$= J_1 (Z_1 Z_2 \otimes [X_1, X_2])$$

$$= Z_1 Z_2 \otimes J([X_1, X_2])$$

$$= Z_1 Z_2 \otimes [J X_1, X_2]$$

$$= [J_1 (Z_1 \otimes X_1), Z_2 \otimes X_2] \qquad (5.6.31a)$$

and likewise

$$= [Z_1 \otimes X_1, J_1 (Z_2 \otimes X_2)] \qquad (5.6.31b)$$

$\mathbb{R}$-linearity implies that

$$J_1([X, y]) = [JX, y] = [X, Jy] \quad (5.6.32)$$

for all $X, y \in \mathbb{C} \otimes L$. Hence the two commuting complex structures J_1 and J_2 on the vector space $\mathbb{C} \otimes L$ lift to two commuting complex structures on the Lie-algebra. Therefore,

$$\begin{aligned} &J_1 \circ J_2([X, y]) \\ &= J_1([J_2 X, y]) = J_1([X, J_2 y]) \\ &= [J_1 J_2 X, y] = [J_1 X, J_2 y] \\ &= [X, J_1 J_2 y] = \cdots \text{ etc.} \end{aligned} \quad (5.6.33)$$

This implies

$$\begin{aligned} P_\pm([X, y]) &= [P_\pm X, y] \\ &= [X, P_\pm y]. \end{aligned} \quad (5.6.34)$$

In particular

$$[P_+ X, P_- y] = [X, \underbrace{P_+ P_-}_{=0} y] = 0 \quad (5.6.35)$$

These equations immediately imply that Kernel$(P_\pm)$ = Image$(P_\mp)$ are both ideals (neccessarily commuting).

Therefore we get a decomposition in analogy to (5.6.24) in the real case

$$(\mathbb{C} \otimes_{\mathbb{R}} L) = (\mathbb{C} \otimes_{\mathbb{R}} L)_+ \oplus (\mathbb{C} \otimes_{\mathbb{R}} L)_- \qquad (5.6.36)$$

where

$$(\mathbb{C} \otimes_{\mathbb{R}} L)_\pm := \{ X \in \mathbb{C} \otimes_{\mathbb{R}} L : J_1(X) = \pm J_2(X) \} \qquad (5.6.37)$$

but now the factors are <u>ideals</u> of the Lie-algebra $\mathbb{C} \otimes_{\mathbb{R}} L$.

The complex case for $(\mathbb{C} \otimes_{\mathbb{R}} L)^{\mathbb{C}}$ is again totally analogous and we get

$$(\mathbb{C} \otimes_{\mathbb{R}} L)^{\mathbb{C}} = (\mathbb{C} \otimes_{\mathbb{R}} L)^{\mathbb{C}}_+ \oplus (\mathbb{C} \otimes_{\mathbb{R}} L)^{\mathbb{C}}_- \qquad (5.6.38)$$

where

$$(\mathbb{C} \otimes_{\mathbb{R}} L)^{\mathbb{C}}_\pm := \{ X \in (\mathbb{C} \otimes_{\mathbb{R}} L)^{\mathbb{C}} : J_2(X) = \pm i X \} \qquad (5.6.39)$$

With respect to the complex structure J on L we see that its extension J_2 to $(\mathbb{C} \otimes_{\mathbb{R}} L)^{\mathbb{C}}$ corresponds to multiplication by $\pm i$. Hence the two ideals are isomorphic to $L^{\mathbb{C}}$ and $\bar{L}^{\mathbb{C}}$ respectively:

$$(\mathbb{C} \otimes_{\mathbb{R}} L)^{\mathbb{C}} = L^{\mathbb{C}} \oplus \bar{L}^{\mathbb{C}}. \qquad (5.6.40)$$