Exercises for the lecture on

Foundations and Applications of Special Relativity

von DOMENICO GIULINI

Sheet 1

Problem 1

Consider n point masses m_a that move along spatial trajectories $t \mapsto \mathbf{x}_a(t) \in \mathbb{R}^3$ under their mutual Newtonian gravitational attraction. Show that

$$\ddot{\mathbf{x}}_a(t) = -\sum_{\substack{b=1\\ b\neq a}}^{n} G m_b \frac{\mathbf{x}_a(t) - \mathbf{x}_b(t)}{\|\mathbf{x}_a(t) - \mathbf{x}_b(t)\|^3} \tag{1}$$

Prove that these equations are in invariant unter any combination of the following operations: 1) temporal translations, 2) spatial translations, 3)... spatial rotations, and 4) Galilei boosts. (Hint: First formulate how these operations act on the trajectories. Note that the independent variable t as well as the dependent variables $\mathbf{x}_a$ are affected.)

Problem 2

Let K denote an inertial frame of reference with respect to which any point in space can be uniquely represented by $\mathbf{x} \in \mathbb{R}^3$. Let K' be any other – generally non-interial – frame of reference with respect to which a point in space is characterised by $\mathbf{x}' \in \mathbb{R}^3$. The relations between these frames is such that for any given point is space we have

$$\mathbf{x} = \mathbf{b}(t) + D(t)\mathbf{x}' \tag{2}$$

Here $\mathbf{b}(t) \in \mathbb{R}^3$ expresses the time-dependent spatial translation and the orthogonal 3×3 matrix $D(t)$ the time dependent spatial rotation of K' against K.

A point-particle moves with respect to the inertial frame K according to Newton's law $\mathbf{F}/m = \ddot{\mathbf{x}}(t)$. Deduce the corresponding equation with respect to K'?

Problem 3

Suppose we identified 3-dimensional euclidean space, which we shall denote by M, with a hyperplane in $\mathbb{R}^4$, of constant zeroth coordinate:

$$M := \left\{(x^0, x^1, x^2, x^3)^\top \in \mathbb{R}^4 : x^0 = 1\right\}. \tag{3}$$

Show that a euclidean motion in M consisting of an orthogonal rotation D followed by a translation $\mathbf{a}$ is then represented by the 4×4 matrix (written in 1+3 block form)

$$g(\mathbf{a}, D) := \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{a} & D \end{pmatrix}. \tag{4}$$

Show that

$$\mathsf{g}(\mathbf{a}_1, \mathsf{D}_1)\, \mathsf{g}(\mathbf{a}_2, \mathsf{D}_2) := \mathsf{g}(\mathbf{a}_1 + \mathsf{D}_1\mathbf{a}_2\,,\, \mathsf{D}_1\mathsf{D}_2)\,. \tag{5}$$

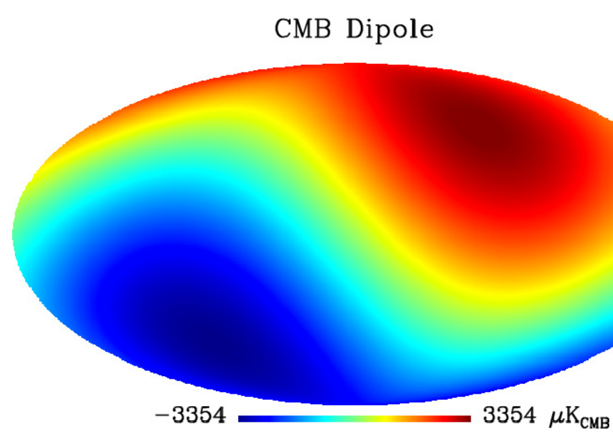
and that $\mathsf{g}(-\mathsf{D}^{-1}\mathbf{a}\,,\, \mathsf{D}^{-1})$ is the inverse of $\mathsf{g}(\mathbf{a}, \mathsf{D})$. Finally show that the subsets of matrices (E_3 denotes the 3×3 unit matrix).

$$\mathsf{T} := \left\{\mathsf{g}(\mathbf{a}, \mathsf{D}) : \mathsf{D} = \mathsf{E}_3\right\}, \tag{6a}$$

$$\mathsf{R} := \left\{\mathsf{g}(\mathbf{a}, \mathsf{D}) : \mathbf{a} = \mathbf{0}\right\}, \tag{6b}$$

are both subgroups, of which T is abelian and invariant (as set) under conjugation, and R is not abelian and not invariant. This means that the euclidean Group contains uncountably many *different* copies of the rotation group (namely R and all conjugates of it by any translation). How do you interpret these different copies?

Problem 4



The picture shows the temperature distribution over the sky of the Cosmic-Microwave-Background-Radiation as measured from Earth. Its mean temperature is $\mathsf{T} = 2.725\,\mathrm{K}$ and shows a dipole anisotropy $\Delta\mathsf{T} = 3.354 \times 10^{-3}\,\mathrm{K}$. It can be explained as being due to the Earth's motion relative to that frame in which the radiation is isotropic (up to $\Delta\mathsf{T}/\mathsf{T} \approx 10^{-5}$). The velocity of the earth relative to that system is then given by $\Delta\mathsf{T}/\mathsf{T} = v/c$, resulting in $v = 369\,\mathrm{km}\cdot\mathrm{s}^{-1}$.

This means that there is an absolute frame of reference throughout the universe with respect to which we can determine our absolute velocity. Discuss whether this disproves the principle of relativity! (Hint: Recall Galileo's example in the *Discorsi* with the ship uniformly sailing on still waters.) Form a definite opinion either pro or con and justify it.