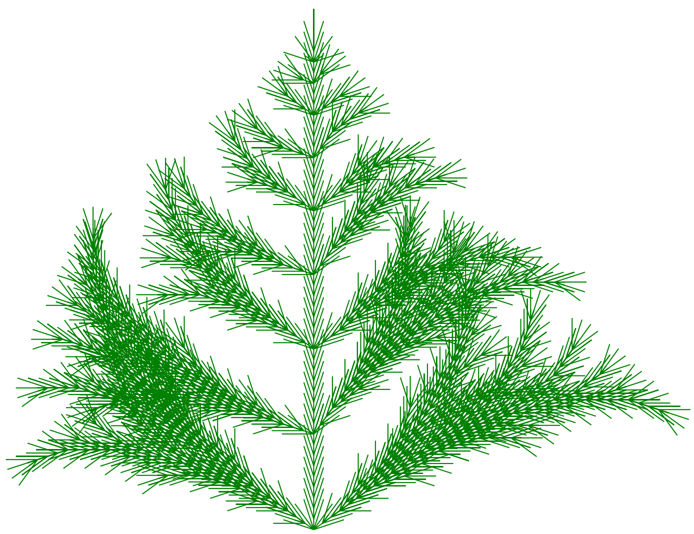
Christmas exercises for the lecture on
Foundations and Applications of Special Relativity

by Domenico Giulini



stay healthy and a happy new year!!!

Sheet 10

Problem 1

A rocket is accelerated by ejecting gas with constant velocity of magnitude **w** (relative to the rest system of the rocket) and constant rate of mass-units per second. We consider only one-dimensional motion in which the rocket moves in free space (no other forces acting; in particular no gravitational force) along the positive z-axis.

In this exercise we wish to derive the special-relativitic version of the Tsiolkovsky equation[1].For this you will need the following three special-relativistic laws: 1) energy conservation; 2) momentum conservation; 3) velocity addition. At each time t think of the total system as consisting of two subsystems: the rocket (system 1) and the gas that has been ejected up to t (system 2). Note that the rest-masses of the rocket, m_1, and of the gas, m_2, are functions of time: the rocket looses mass and the gas gains mass due to ejection.

The whole process is described with respect to one and the same inertial system in which the rocket is initially at rest at the origin and then starts its thrusters at $t = 0$ ejecting gas in the negative z-direction, as a result of which with the rocket accelerates in the positive z-direction. In the following we use the abbreviations $\beta := w/c$, $\beta_{1,2} := v_{1,2}/c$, and γ resp. $\gamma_{1,2}$ for the corresponding γ-factors.

[1] After the Russian rocket scientist Konstantin Tsiolkovsky (1857–1935).

1. The energy and momentum gained by the rocket in a time intervall dt must equal minus the energy and momentum gained by the gas in the same time interval. Show that this leads to

$$d(m_1\gamma_1) = -\gamma_2\, dm_2\,, \tag{1a}$$
$$d(m_1\gamma_1\beta_1) = -\gamma_2\beta_2\, dm_2\,. \tag{1b}$$

   Show further that

$$\beta_2 = \frac{\beta_1 - \beta}{1 - \beta_1\beta}\,. \tag{2}$$

2. Take equation (1b) (momentum conservation) and eliminate in it dm_2 via (1a) and β_2 via (2). This leads to an equation involving only β_1, m_1, their differentials, and the constant β. Separation of variables allows you to integrate it either as $m_1(\beta_1)$ or as $\beta_1(m_1)$. State both forms and use the initial condition that $\beta = 0$ for $m_1 = M$ (initial mass). Show that

$$\gamma_1(m_1) = \frac{1}{2}\left[\left(\frac{m_1}{M}\right)^{\beta} + \left(\frac{m_1}{M}\right)^{-\beta}\right] \tag{3}$$

3. What γ-factor would you have to reach in order to travel to our neighbouring galaxy, Andromeda, within a human lifespan (say 80 years)? How much of your spaceship would you have to burn at least in order to achieve that? If at maximal γ you meet a grain of cosmic dust of mass 1 μg that gets stuck in your "windscreen", how much kinetic energy will it deposit? What speed would a Rolls-Royce luxury limousine of mass 3 tons have to assume in order to reach that kinetic energy?

Problem 2

This exercise is almost identical to Problem 5 on Sheet 9, only the initial conditions are different.

A particle of charge e moves in a constant electric field $\mathbf{E} = E\mathbf{e}_x$ pointing in x-direction. Its strength E is constant. The Lagrangian is (in this problem a dot denotes the t-derivative)

$$L(\mathbf{x}, \dot{\mathbf{x}}) = -mc^2\sqrt{1 - \frac{\dot{\mathbf{x}}^2}{c^2}} + eEx\,. \tag{4}$$

1. Solve the equations of motion with initial position $\mathbf{x}(t = 0) = \mathbf{0}$ and initial velocity $\dot{\mathbf{x}}(t = 0) = v_0\mathbf{e}_y$ with $v_0 > 0$. Show that the graph of the spatial trajectory $x(y)$ is a catenary. What does the catenary turn into in the non-relativistic limit $1/c \to 0$?

2. Compare the coordinate time t and the proper time τ it takes for the particle to move from $x = 0$ at $t = 0$ to a fixed value $x = h$.

Hint. As for Problem 5 on Sheet 9, a first integration of the Euler-Lagrange-Equation with the given initial data is easy to do. You obtain two equations $\gamma\dot{x} = C_1 t$ and $\gamma\dot{y} = C_2$ with some constants $C_{1,2}$. But this is not yet ready for integration because γ involves $\dot{x}$ as well as $\dot{y}$. One strategy you can follow is to square the second equation and use $\dot{x} = \dot{y}(C_1/C_2)t$ to eliminate $\dot{x}$. The resulting equation only contains $\dot{y}$ and t which you can integrate. The rest follows...

Problem 3

Let $u_1 \in V$ and $u_2 \in V$ be two states of motion (future oriented unit timelike vectors).

1. show that the modulus of the relative velocity between them (in units of c) is (a dot denotes the Minkowski scalar product)

$$\beta = \frac{\sqrt{(u_1 \cdot u_2)^2 - 1}}{u_1 \cdot u_2}. \tag{5}$$

2. Let u be a third state of motion which we complete to an orthonormal basis $\{e_0, e_1, e_2, e_3\}$, where $e_0 = u$. Relative to that basis the velocities of u_1 and u_2 are then described by the components $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, i.e. $u_1 = \gamma_1(e_0 + \beta_1^k e_k)$ and $u_2 = \gamma_2(e_0 + \beta_2^k e_k)$, where the sum over k ranges from 1 to 3 and where $\gamma_{1,2}$ is the gamma-factor for $\boldsymbol{\beta}_{1,2}$. Show that the modulus of the relative velocity between u_1 and u_2 is then given by

$$\beta = \frac{\sqrt{(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2)^2 - (\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2)^2}}{1 - \boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2}. \tag{6}$$

   Note the following: $\beta_{1,2}$ refer to u, i.e. are velocities "measured" in the rest frame of u. But β refers to the velocity of either u_2 relative to u_1, in which case it is measured by u_1, or u_1 relative to u_2, in which case it is measured by u_2. There is also a notion of "relative velocity of u_1 to u_1 measured by u", which is different from that considered here.

Problem 4

Consider a timelike curve $\gamma : \mathbb{R} \to M$ in Minkowski space. Its derivative with respect to its proper time τ is denoted by a dot. We consider the first, second, and third proper-time derivatives $\dot{\gamma}$, $\ddot{\gamma}$, and $\dddot{\gamma}$, respectively. $\dot{\gamma} = v$ is also called the four-velocity and $\ddot{\gamma} = a$ the four-acceleration of the curve γ.

1. Show that the four-acceleration is always perpendicular to the four-velocity, i.e.

$$\dot{\gamma} \cdot \ddot{\gamma} = 0. \tag{7}$$

2. The curve γ is said to be of **constant acceleration** iff

$$P^{\perp}_{\dot{\gamma}}(\dddot{\gamma}) := \dddot{\gamma} - \dot{\gamma}\,\frac{\dot{\gamma} \cdot \dddot{\gamma}}{c^2} = 0 \tag{8}$$

where $P^{\perp}_{\dot\gamma}$ is the projection into the orthogonal complement of $\dot\gamma$ (note that $\dot\gamma^2 = c^2$ because the dot refers to proper time, which is c^{-1} times proper length). State in words in what sense (8) is a reasonable definition of “constant acceleration” and then show that it is equivalent to

$$\dddot\gamma = \omega^2\,\dot\gamma \tag{9a}$$

where ω is a real, non-negative constant given by

$$\omega := \sqrt{-\frac{\ddot\gamma^2}{c^2}}\,. \tag{9b}$$

Hint: First recall (7) and show that (8) implies that $\ddot\gamma^2$ is constant along the curve.

3. Integrate (9a) and show that the most general solution is given by

$$\gamma(\tau) = p + \Big(\ell_1\;\exp(\omega\tau) - \ell_2\;\exp(-\omega\tau)\Big)\,, \tag{10a}$$

where p is some point in M and ℓ_1, ℓ_2 are two future-pointing lightlike vectors in V satisfying

$$\ell_1\cdot\ell_2 = \frac{c^2}{2\,\omega^2}\,. \tag{10b}$$

4. Show that

$$e_0 := \frac{\omega}{c}(\ell_1+\ell_2)\,,\quad e_1 := \frac{\omega}{c}(\ell_1-\ell_2) \tag{11}$$

is a orthonormal ($e_0\cdot e_0 = -e_1\cdot e_1 = 1$, $e_0\cdot e_1 = 0$) basis of $\mathrm{span}\{\ell_1,\ell_2\}$ in terms of which (10a) reads

$$\gamma(\tau) = p + \frac{c}{\omega}\Big(\sinh(\omega\tau)\,e_0 + \cosh(\omega\tau)\,e_1\Big) \tag{12}$$

showing that we are dealing with a hyperbolic motion in the timelike affine plane $p + \mathrm{span}\{e_0,e_1\} = p + \mathrm{span}\{\ell_1,\ell_2\}$ of Minkowski space the constant acceleration of which is $\sqrt{-\ddot\gamma^2} = \omega c$.

Problem 5

Let us look at the constant-acceleration motion in a coordinate-based language. With respect to adapted affine coordinates the $\{x^0,x^1,x^2,x^3\}$ with $x^0 = ct$ and t as curve parameter a timelike curve in Minkowski space is analytically described by ($\alpha = 0,\cdots,3$)

$$t \mapsto x^\alpha(t) := \big(ct\,,\,\mathbf{x}(t)\big) \tag{13}$$

1. Show that the derivative with respect to eigentime τ and coordinate time t are related by

$$\frac{d}{d\tau} = \gamma(t)\frac{d}{dt} \tag{14}$$

where $\gamma(t) = 1/\sqrt{1-\mathbf{x}'^2(t)/c^2}$. In the following we denote derivatives with respect to t by a prime and with respect to τ by a dot and abbreviate

$$\mathbf{x}' = \mathbf{v}\,,\qquad \mathbf{x}'' = \mathbf{a}\,,\qquad \mathbf{x}''' = \mathbf{b}\,. \tag{15}$$

2. Show
$$\dot{x}^\alpha = \gamma(c, \mathbf{v}) \quad \text{and} \quad \gamma' = \gamma^3 \frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\,. \tag{16}$$
and further
$$\begin{aligned} \ddot{x}^\alpha &= \gamma'\dot{x}^\alpha + \gamma^2(0, \mathbf{a})\,, & (17)\\ \dddot{x}^\alpha &= (\gamma\gamma')'\dot{x}^\alpha + \gamma^2(0\,,\, 3\gamma'\mathbf{a} + \gamma\mathbf{b})\,. & (18) \end{aligned}$$

3. The condition (8) of constant acceleration is equivalent to
$$\dddot{x}^\alpha - \dot{x}^\alpha \frac{\eta_{\mu\nu}\,\dot{x}^\mu\,\dddot{x}^\nu}{c^2} = 0\,. \tag{19}$$
Show that applied to (18) this is equivalent to
$$3\gamma'\mathbf{a} + \gamma\mathbf{b} = 0\,, \tag{20}$$
that is
$$(\gamma^3\mathbf{a})' = 0\,, \tag{21}$$
which is the coordinate-equivalent to (8).

4. Finally, denote by $\mathbf{a}_\parallel$ and $\mathbf{a}_\perp$ the components parallel and perpendicular to $\mathbf{v}$ of $\mathbf{a}$. Show that
$$(\gamma\mathbf{v})' = \gamma^3\mathbf{a}_\parallel + \gamma\mathbf{a}_\perp\,, \tag{22}$$
and conclude that if (21) is once integrated to $\gamma^3\mathbf{a} = \mathbf{g}$, where $\mathbf{g} \in \mathbb{R}^3$ is constant, and the initial velocity is parallel to $\mathbf{g}$, then the motion satisfies
$$(\gamma\mathbf{v})' = \mathbf{g} \tag{23}$$
and can be integrated as in Problem 5 of Sheet 9 resulting in a hyperbolic motion in the plane containing the t axis and the spatial axis in $\mathbf{g}$ direction.

   But even if the initial velocity is not parallel to $\mathbf{g}$, we still know from the previous Problem 4 that *any* solution to (21) results in a hyperbolic motion within a single timelike plane. Can that also be deduced directly from (23)? Note that in that case (23) is not valid. Is the solution to Problem 2 above of constant acceleration?