Exercises for the lecture on

Foundations and Applications of Special Relativity

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Sheet 11

Problem 1

Consider the electromagnetic Coulomb field

$$\mathbf{E}(t,\mathbf{x}) = \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{x}}{\|\mathbf{x}\|^3} \qquad \mathbf{B}(t,\mathbf{x}) = \mathbf{0}\,, \tag{1}$$

1. Derive the expressions $\mathbf{E}'(t,\mathbf{x})$ and $\mathbf{B}'(t,\mathbf{x})$ for the field obtained from boosting (1) with velocity $\mathbf{v}$ in x^1-direction. Draw the field-lines for both fields at $t = 0$.
2. A possible four-potential A_α for the (1) is $A_0 = \phi/c = Q/(4\pi\varepsilon_0 c)(1/\|\mathbf{x}\|)$ and $A_m = 0$ for $m = 1, 2, 3$. What is the four-potential for the boosted field? Draw the equipotential-surfaces for the boosted A'_0.

Problem 2

We consider the energy-momentum tensor T as element in $V \otimes V^*$, that is, as endomorphism on V. It is then assumed to be symmetric with respect to the Minkowski inner product η. T is said to be "diagonalisable" if an orthonormal basis for V exists each element of which is an eigenvector of T.

- Prove that T is diagonalisable iff it has a timelike eigenvector.

Problem 3

The four-vector potential of a plane-wave solution to Maxwells equations is

$$A_\mu(x) = a_\mu \cos(k_\alpha x^\alpha + \varphi_\mu)\,, \tag{2}$$

where $k \in V$ is the "four-wavevector" which is light-like ($k_\alpha k^\alpha = 0$), where the amplitiude a_μ and phases φ_μ are constant and the amplitudes are perpendicular to the wave-vector, $a_\mu k^\mu = 0$.

- Calculate the energy-momentum tensor for the corresponding electromagnetic field. Is it diagonalisable?

Problem 4

Let V be a real four-dimensional vector-space with Minkowski inner product. We endow V with an orientation. We define a "volume form" $\varepsilon \in \bigwedge^4 V^*$ as follows: Let $\{e_0, e_1, e_2, e_3\}$ be an oriented orthonormal basis with corresponding dual basis $\{\theta^0, \theta^1, \theta^2, \theta^3\}$; then

$$\varepsilon = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 = \frac{1}{4!}\, \varepsilon_{\alpha\beta\gamma\delta} \theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma \wedge \theta^\delta \,. \tag{3}$$

Note that the component $\varepsilon_{\alpha\beta\gamma\delta}$ is +1/-1 iff $(\alpha\beta\gamma\delta)$ is an even/odd permutation of (0123), and zero otherwise.

1. Show that ε does not depend on which oriented orthonormal basis $\{e_0, e_1, e_2, e_3\}$ is chosen. In other words: The ε as defined in (3) by means of a specific basis is, in fact, invariant under orientation-preserving Lorentz transformations.

2. Let $\varepsilon^{\alpha\beta\gamma\delta}$ be the index-raised version of $\varepsilon_{\alpha\beta\gamma\delta}$. Show that

$$\varepsilon^{\alpha\beta\gamma\delta} = -\varepsilon_{\alpha\beta\gamma\delta} \tag{4}$$

and further that (this is a "once in a lifetime" exercise; your time has come ...)

$$\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\lambda\mu\nu\sigma} = -\,0!\; \delta^{\alpha\beta\gamma\delta}_{\lambda\mu\nu\sigma}\,, \tag{5a}$$

$$\varepsilon^{\alpha\beta\gamma\sigma}\varepsilon_{\lambda\mu\nu\sigma} = -\,1!\; \delta^{\alpha\beta\gamma}_{\lambda\mu\nu}\,, \tag{5b}$$

$$\varepsilon^{\alpha\beta\nu\sigma}\varepsilon_{\lambda\mu\nu\sigma} = -\,2!\; \delta^{\alpha\beta}_{\lambda\mu}\,, \tag{5c}$$

$$\varepsilon^{\alpha\mu\nu\sigma}\varepsilon_{\lambda\mu\nu\sigma} = -\,3!\; \delta^{\alpha}_{\lambda}\,, \tag{5d}$$

$$\varepsilon^{\lambda\mu\nu\sigma}\varepsilon_{\lambda\mu\nu\sigma} = -\,4! \tag{5e}$$

where $0! = 1! = 1$ are just written down to make it look nicer (more systematic) and we used the abbreviation

$$\delta^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n} := n!\; \delta^{\alpha_1}_{[\beta_1} \cdots \delta^{\alpha_n}_{\beta_n]} := \sum_{g\in S_n} \mathrm{sgn}(g)\; \delta^{\alpha_1}_{g(\beta_1)} \cdots \delta^{\alpha_n}_{g(\beta_n)}\,. \tag{6}$$

Here S_n is the symmetric group (also called the group of all permutations) of a set of n elements (here the n indices).

Some explanation

The following exercises use the results of the previous Problem. They can be done independently by just assuming the results and need not be proven once more. They also use the concept of Hodge-Duality, which is this: In n dimensions, the volume form ε defines an isomorphism of the vector-spaces $\bigwedge^k V^*$ and $\bigwedge^{n-k} V^*$. In 4 dimensions, to which we restrict attention to, it defines in particular a self-isomorphism of $\bigwedge^2 V^*$, given in components by

$$\star\, F_{\mu\nu} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}F^{\mu\nu} \tag{7}$$

Equation (5c) then implies

$$\star \circ \star = -\,\mathrm{id}\,, \tag{8}$$

as you may check.

Problem 5

Let n be a "state of motion", that is, a timelike normalised vector. By $i_n F$ we denote the one-form that results from contracting n with F. In index-form: $(i_n F)_\beta = n^\alpha F_{\alpha\beta}$. Let further $n^\flat$ be the corresponding one form to n (index-lowered version of n).

- Prove the following identity

$$F = n^\flat \wedge i_n F - \star(n^\flat \wedge i_n \star F)\,. \tag{9}$$

This is said to be a decomposition of F into the "electric part" $i_n F$ and "magnetic part" $-i_n \star F$ with restpect to n (the state of motion). Check this interpretation with our labelings of the components of the Faraday-tensor.

A useful notation

In the following we save notation and raise the first index on two forms, i.e. $F_{\alpha\beta} \mapsto F^\alpha{}_\beta := \eta^{\alpha\mu} F_{\mu\beta}$ and denote products by $(FG)^\alpha{}_\beta = F^\alpha{}_\mu G^\mu{}_\beta$ and accordingly $(F^2)^\alpha{}_\beta = F^\alpha{}_\mu F^\mu{}_\beta$. We then drop writing the indices in our equations.

Exercise 6

This exercise contains a string of useful results concerning the algebraic properties of the Faraday-tensor and its associated energy-momentum tensor.

1. Show that the characteristic polynomial of the endomorphism F is given by

$$P_F(\lambda) := \det\big(F - \lambda\, \mathrm{id}_V\big) := \lambda^4 - \tfrac{1}{2}\lambda^2\, \mathrm{Trace}(F^2) + \det(F)\,. \tag{10}$$

   Hint: Use the fact that the determinant of an endomorphism X can be written as

$$\det(X) = -\frac{1}{4!}\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\lambda\mu\nu\sigma} X^\lambda{}_\alpha X^\mu{}_\beta X^\nu{}_\gamma X^\sigma{}_\delta \tag{11}$$

   and use (5a) (here applied to $X = F - \lambda\, \mathrm{id}_V$.

2. Prove that

$$FG - (\star G)(\star F) = \frac{1}{2}\mathrm{Trace}(FG)\, \mathrm{id}_V\,. \tag{12}$$

   and from that, by specialisation,

$$F^2 - \star F^2 = 2I_1\, \mathrm{id}_V\,, \tag{13a}$$

$$F \star F = \star F\, F = I_2\, \mathrm{id}_V\,, \tag{13b}$$

$$F^4 - 2I_1\, F^2 - I_2^2\, \mathrm{id}_V = 0\,, \tag{13c}$$

   where we have used the following abbreviation:

$$I_1 := \frac{1}{4}\mathrm{Trace}(F^2) \quad \text{and} \quad I_2 := \frac{1}{4}\mathrm{Trace}\big(F \star F\big)\,. \tag{14}$$

   Hint: Write out $(\star G)(\star F)$ using the definition (7) and use (5) to deal with the multiplication of two ε's.

3. Think of F as being the electrodynamic Faraday-tensor. Its "energy-momentum tensor" T, also thought of as endomorphism (one index up one down) is just trace-free part of F^2 (here we drop the constant SI-constant $/\mu$ on the right-hand side):

$$T = F^2 - \frac{1}{4}\mathrm{Trace}(F^2) = F^2 - I_1\,\mathrm{id}_V\,. \tag{15}$$

Use (12) to prove

$$T = \tfrac{1}{2}(F^2 + \star F^2) \tag{16}$$

and (13c) to prove the *Rainich-identity*[1]

$$T^2 = (I_1^2 + I_2^2)\,\mathrm{id}_V\,. \tag{17}$$

4. Use (16) to prove that T is invariant under *duality-transformations*

$$\begin{pmatrix} F \\ \star F \end{pmatrix} \mapsto \begin{pmatrix} F_\theta \\ \star F_\theta \end{pmatrix} := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F \\ \star F \end{pmatrix}\,. \tag{18}$$

Show also that this can be expressed by defining the complex field-strength

$$\omega := F + i \star F \tag{19}$$

by proving (using (13c)) that

$$T = \tfrac{1}{2}\omega\overline{\omega}\,. \tag{20}$$

Then (18) is just $\omega \mapsto \omega_\theta := \exp(-i\theta)\,\omega$.

How does the duality transformation read in terms of **E** and **B**?

5. Show that

$$\det(F) = \det(\star F) = -I_2^2\,. \tag{21}$$

Hint: To prove the first equality compare (13c) with (10) and use the Cayley-Hamitlton-Theorem (i.e. $p_A(A) = 0$) . Use (13b) for the second equality.

6. Prove that eigenvalues λ of F satisfy

$$\lambda^2 = I_1 \pm \sqrt{I_1^2 + I_2^2}\,. \tag{22}$$

7. Prove that any trace-polynomial

$$\mathrm{Trace}(XY\cdots Z)\,, \tag{23}$$

with entries X, Y, Z either F or $\star F$, is a polynomial in I_1 and I_2.

[1] After George Yuri Rainich (1886-1968), born as George Yuri Rabinovich, who was a Ukrainian mathematician who emigrated to the USA and greatly influenced mathematical physics.