Exercises for the lecture on

Foundations and Applications of Special Relativity

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Sheet 12

Problem 1

A point particle of mass m and charge q moves along a future-oriented timelike curve $\tau \to z(\tau) \in M$, where we take the parameter τ to be the eigentime; hence $\dot{z}^2 = c^2$, where $\dot{z} := dz/d\tau$. The components of J^α of the electric four-current density and the components of the energy-momentum tensor are now distributions on M and given by

$$J^\alpha(x) := qc \int d\tau\, \delta^{(4)}\big(x - z(\tau)\big)\, \dot{z}^\alpha(\tau)\,, \tag{1}$$

$$T^{\alpha\beta}(x) := mc \int d\tau\, \delta^{(4)}\big(x - z(\tau)\big)\, \dot{z}^\alpha(\tau)\dot{z}^\beta(\tau)\,. \tag{2}$$

1. Justify these expressions.
2. Show that the expression (1) could just as well have been written in terms of any other parameter λ instead of τ.
3. Prove that $\nabla_\alpha J^\alpha = 0$ (as a distribution).
4. Prove that $\nabla_\alpha T^{\alpha\beta} = 0$ (as a distribution) iff $\ddot{z} = 0$, i.e. if the curve is a straight line.

Problem 2

1. Prove that the relativistic Maxwell-Equations for a given source J^α are the Euler-Lagrange-Equations for the following action functional:

$$S_f[A] = \int d^4x\, \frac{1}{c} \left\{ -\frac{1}{4\mu_0}\, F_{\alpha\beta}(x) F^{\alpha\beta}(x) - A_\alpha(x)\, J^\alpha(x) \right\}\,, \tag{3}$$

   where the field to be varied is A, on which F depends via $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$.

   Hint: If $\mathscr{L}$ denotes the integrand of the action functional (called the Lagrange density), the Euler-Lagrange-Equations are

$$\nabla_\alpha \left(\frac{\partial \mathscr{L}}{\partial(\nabla_\alpha A_\beta)} \right) - \frac{\partial \mathscr{L}}{\partial A_\beta} = 0\,. \tag{4}$$

2. Why did we choose the overall sign and factor the way we did? (The Euler-Lagrange-Equations are insensitive to that.)

3. A relativistic particle of mass m is described by a future-oriented timelike worldline $\lambda \to z(\lambda)$, where now λ is an arbitrary parameter. As we know, its action is

$$S_p[z] = -mc \int d\lambda \, \sqrt{z'_\alpha(\lambda) z'^\alpha(\lambda)} \tag{5}$$

If the particle carries a charge q, it has a four-current density given by (1) (with λ replacing τ). Suppose the particle moves in an electromagnetic field represented by A, then its interaction with the field contributes a term according to (3). Show that the total action of the particle, now including its interaction with the field, is

$$S_{p+int} = \int d\lambda \Big\{ -mc\sqrt{z'_\alpha(\lambda) z'^\alpha(\lambda)} - q\, A_\alpha\big(z(\lambda)\big)\, z'^\alpha(\lambda) \Big\} \tag{6}$$

and that its Euler-Lagrange-Equations are

$$m\ddot{z}^\alpha(\tau) = q\, F^\alpha{}_\beta\big(z(\tau)\big)\, \dot{z}^\beta(\tau)\,, \tag{7}$$

where a dot is again the derivative with respect to the eigentime parameter τ which – we recall – obeys $c\,d\tau = \sqrt{z'^\alpha(\lambda) z'_\alpha(\lambda)}\; d\lambda$.

Hint: In this case, if L denotes the integrand for the integral (6) (the Lagrange function), the Euler-Lagrange-Equations are

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial z'^\beta}\right) - \frac{\partial L}{\partial z^\beta} = 0\,. \tag{8}$$

Problem 3

The Abraham-Lorentz-Equation in “ordinary” electrodynamics is meant to describe the spatial curve $t \mapsto \mathbf{z}(t)$ of a slowly (i.e. $\|\dot{\mathbf{z}}\|/c \ll 1$) moving point-particle of mass m and charge q under the action of an external electromagnetic field $(\mathbf{E}, \mathbf{B})$ and the particle’s own radiation-reaction force. It reads:

$$m\ddot{\mathbf{z}} = q\big(\mathbf{E} + \dot{\mathbf{z}} \times \mathbf{B}\big) + \sigma\, m\, \dddot{\mathbf{z}}\,, \tag{9}$$

where σ is a constant with the physical dimension of time:

$$\sigma = \frac{\mu_0 q^2}{6\pi c m}\,. \tag{10}$$

1. Show that σ can be rewritten as

$$\sigma = \frac{4}{3} \cdot \frac{q^2}{8\pi\varepsilon_0 m c^2} \cdot \frac{1}{c} \tag{11}$$

   and use that form to interpret σ as a travel-time for light over a certain distance. What is that distance? What is its numerical value for the electron?

2. The relativistic generalisation of the the Abraham-Lorentz-Equation is the Lorentz-Dirac-Equation, which is *not* restricted to small velocities. It reads in components for the worldline $\tau \mapsto z^\alpha(\tau)$, parametrised by the eigentime τ,

$$m\,\ddot{z}^\alpha = q\, F^\alpha{}_\beta \dot{z}^\beta + m\sigma\left(\dddot{z}^\alpha + \frac{\ddot{z}^\beta \ddot{z}_\beta}{c^2}\, \dot{z}^\alpha\right)\,. \tag{12}$$

Here a dot now indicates the proper-time derivative. Argue that this is indeed the "obvious" generalisation of the Abraham-Lorentz equation (9). In particular, argue why the second factor in the bracket on the right-hand side is necessary and that without it the equation would certainly not capture the physically intended meaning and have almost no solutions. (Hint: Multiply the equation with $\dot{z}_\alpha$ and recall that a four-acceleration $\ddot{z}$ is always perpendicular to the four-velocity $\dot{z}$.) What is the geometric interpretation of the sum the two terms in the bracket? (Hint: Compare Problem 4 on Sheet 10).

3. In Problem 5 of Sheet 9 we discussed the solution for the equation of motion for a point particle in a constant electric field, given that the particle was initially at rest. In Problem 5 (part 4) of Sheet 10 we identified this solution to be of constant acceleration. Prove that this particular solution also solves the Lorentz-Dirac equation (for the same constant electric field).

4. Without external field the Lorentz-Dirac-Equation van be written as

$$\dot{u}^\alpha = \sigma\Big(\ddot{u}^\alpha + \dot{u}^\beta \dot{u}_\beta \, u^\alpha\Big)\,, \tag{13}$$

   where we set $u^\alpha := \dot{z}^\alpha / c$ so that $u^\alpha u_\alpha = 1$. It follows from this equation that any solution remains in the timelike plane spanned by the initial vectors u and $\dot{u}$. Let this plane be spanned by the orthonormal vectors e_0 (timelike) and e_1 (spacelike). Then the components u^0 and u^1 of the vector $u = u^0 e_0 + u^1 e_1$, which obey $(u^0)^2 - (u^1)^2 = 1$ may be written in terms of the rapidity $\rho(\tau)$ as follows

$$u^0 = \cosh(\rho) \quad \text{and} \quad u^1 = \sinh(\rho)\,. \tag{14}$$

   Show that (13) is now equivalent to the *linear* equation

$$\ddot{\rho} = \dot{\rho}/\sigma\,. \tag{15}$$

   Use this to determine the most general solution for $\rho(\tau)$ and hence for $z(\tau)$. Show that in addition to the inertial motions (straight worldlines) you get accelerated solutions with exponentially growing rapidity.

Problem 4

This is problem 6 of sheet 11 reloaded.