Exercises for the lecture on

Foundations and Applications of Special Relativity

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Sheet 2

Problem 1

Consider two point masses m_1, m_2 with interaction potential U that only depends on the mutual distance $r := \|\mathbf{x}_1 - \mathbf{x}_2\|$. Here $\mathbf{x}_{1,2}$ are the coordinates with respect to an inertial system.

1. Write down the Lagrangian $L = T - U$ in terms of the $\mathbf{x}_{1,2}$ and derive the corresponding Euler-Lagrange equations.

2. Transform the Lagrangrian to relative- and centre-of-mass coordinates, defined by

$$\mathbf{r} := \mathbf{x}_1 - \mathbf{x}_2 \,, \tag{1a}$$

$$\mathbf{R} := \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2} \,, \tag{1b}$$

   also using the "reduced mass" $\mu := m_1 m_2/(m_1 + m_2)$. Derive the Euler-Lagrange equations once more in these coordinates.

3. Calculate the canonical momenta and write down the Hamiltonian.

4. Now apply an active Galilean boost transformation to the particle trajectories,

$$\mathbf{x}_a(t) \to \mathbf{x}'_a(t) := \mathbf{x}_a(t) + \mathbf{v}t \tag{2}$$

   and use that to replace $\mathbf{x}_a(t)$ in the original Langrangian by $\mathbf{x}'_a(t) - \mathbf{v}t$. Do the same for the centre-of-mass coordinates.

5. Show that the resulting Lagrangian is *not* that obtained by simply replacing $\mathbf{x}_a(t)$ with $\mathbf{x}'_a(t)$, but that the resulting Euler-Lagrange equations *are* that obtained by replacing $\mathbf{x}_a(t)$ with $\mathbf{x}'_a(t)$. What is the mathematical reason for that?

6. How does the Hamiltonian change under (2)?

7. Does Noether's theorem apply in this case? If so, what is the associated conserved quantity?

Problem 2

Consider a rigid cube of side-length ℓ whose centre moves along the positive x-axis with constant velocity v. The normals of the faces of the cube are always parallel to the x, y, z axes. An observer is located on the negative y axis at a distance $D \gg \ell$. The cube is illuminated and the observer takes a snapshot with a camera of ideally zero exposure time. The snapshot is taken at a time at which the centre of the cube appears at the coordinate origin on the snapshot. The velocity of light is assumed to be given by the constant $c \gg v$ in all directions.

How would the cube look like on the snapshot.

Hint: Our assumption of zero exposure time implies that all light-rays originating from parts of the cube have to arrive simultaneously at the observer. This means that the parts of the cube further away from the observer have to emit the light at an earlier time than those closer to the observer. But at earlier times the cube had a different position. This leads to the visual effects that you are asked to describe. Our assumptions $D \gg \ell$ and $c \gg v$ means that to leading order in approximation that you may assume all light rays reaching the observer to be parallel.

Problem 3

Let $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field.

1. State the law according to which it transforms under active spatial rotations.
2. Show that the equations

$$\boldsymbol{\nabla} \cdot \mathbf{V} = 0 \quad \text{and} \quad \boldsymbol{\nabla} \times \mathbf{V} = \mathbf{0} \tag{3}$$

   permit rotations as symmetries in the sense that rotations map solutions to solutions.

3. Derive the form of the most general rotation-invariant vector field and find amongst them all solutions to the first equation (3) and also all solutions to the second equation.