Exercises for the lecture on

Foundations and Applications of Special Relativity

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Sheet 3

Problem 1

In the lecture we wrote down a Lorentz boosts with respect to inertial coordinates (ct, x, y, z). It reads (interpreted actively)

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \longmapsto B(\beta \mathbf{e}_x) \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{1a}$$

where

$$B(\beta \mathbf{e}_x) := \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1b}$$

and

$$\gamma := \frac{1}{\sqrt{1-\beta^2}}\,, \qquad \beta := \frac{v}{c}\,. \tag{1c}$$

1. Show that $B(\beta \mathbf{e}_x)$ is a symmetric and positive-definite linear map in $\mathbb{R}^4$ with respect to the standard euclidean inner product, the eigenvalues of which are

$$\lambda_{1,2} = \gamma \pm \sqrt{\gamma^2 - 1} = \sqrt{\frac{1 \pm \beta}{1 \mp \beta}}\,, \qquad \lambda_{3,4} = 1\,, \tag{2}$$

   (note that $\lambda_2 = \lambda_1^{-1}$) and the normalised eigenvektors of which are

$$\mathbf{E}_{1,2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \\ 0 \\ 0 \end{pmatrix}\,, \qquad \mathbf{E}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\,, \qquad \mathbf{E}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\,. \tag{3}$$

2. A spatial rotation with $D \in SO(3)$ is represented, again with respect to the standard basis in $\mathbb{R}^4$, by the (4×4) matrix (written in 1+3 form)

$$R(D) = \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & D \end{pmatrix}\,. \tag{4}$$

   Show that

$$R(D) B(\beta \mathbf{e}_x) R(D^{-1}) = B(\beta D \mathbf{e}_x), \tag{5}$$

and that, consequently, a boost in a general direction $\mathbf{n} = D\mathbf{e}_x$ is given by

$$B(\beta\mathbf{n}) = \begin{pmatrix} \gamma & \beta\gamma\mathbf{n}^\top \\ \beta\gamma\mathbf{n} & E_3 + (\gamma-1)\mathbf{n}\otimes\mathbf{n}^\top \end{pmatrix} . \tag{6}$$

Here E_3 denotes the unit (3×3) matrix and $\mathbf{n}\otimes\mathbf{n}^\top$ is the (3×3)-matrix with components $n_a n_b$. Hence a Lorentz boost in the direction $\mathbf{n}$ can also be written

$$ct \mapsto \gamma\big(ct + \beta(\mathbf{n}\cdot\mathbf{x})\big)\,, \tag{7a}$$

$$\mathbf{x} \mapsto \mathbf{x} + (\gamma-1)\mathbf{n}(\mathbf{n}\cdot\mathbf{x}) + \beta\gamma\,\mathbf{n}ct \tag{7b}$$

3. Show that (7) also follows directly from (1) by writing $\mathbf{x} = \mathbf{x}_\parallel + \mathbf{x}_\perp$, where $\mathbf{x}_\parallel$ and $\mathbf{x}_\perp$ correspond to the components parallel and perpendicular to the boost direction $\mathbf{e}_x$.

4. Show that a general Boost transformation leaves a 2-dimensional plane in $\mathbb{R}^4$ pointwise fixed and transforms the orthogonal plane (with respect to the euclidean inner product) non-trivially into itself with only fixed-point being the origin. Give a geometric description of the orbits of that action.

Problem 2

Let $M \in GL(\mathbb{R}^n)$ be a general invertible $(n\times n)$-matrix with real entries. Show that it has a unique so-called "polar decomposition"

$$M = PO \tag{8}$$

into an orthogonal matrix $O \in O(3)$ and a symmetric and positive-definite matrix P (written in the order just stated).

Hint: Consider $MM^\top$ and prove that it is symmetric and positive definite. Argue that there is a positive square-root $P := \sqrt{MM^\top}$ which is also symmetric and positive definite, so that $P^2 = MM^\top$. Then define $O := P^{-1}M$ and prove that it is orthogonal. This proves existence. For uniqueness assume $M = P_1O_1 = P_2O_2$ with $P_{1,2}$ symmetric and positive-definite and $O_{1,2}$ orthogonal. Show that then $P_1 = P_2$ and $O_1 = O_2$ follow.

Problem 3

This problem is partly a repetition of Problem 3 on Sheet 2 which we did not yet come to discuss so far. Let $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field.

1. Show that the equations

$$\boldsymbol{\nabla}\cdot\mathbf{V} = 0 \quad \text{and} \quad \boldsymbol{\nabla}\times\mathbf{V} = \mathbf{0} \tag{9}$$

permit rotations as symmetries in the sense given on Sheet 2.

2. Derive the form of the most general rotation-invariant vector field and find amongst them all solutions to the first equation (9) and also all solutions to the second equation.