Exercises for the lecture on

Foundations and Applications of Special Relativity

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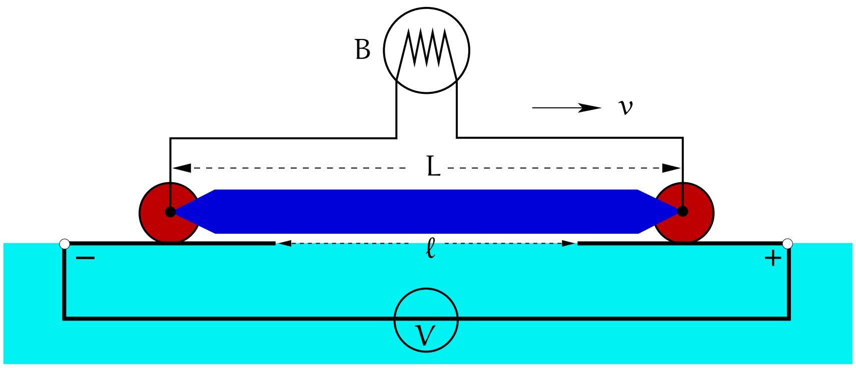
Sheet 4

Problem 1



The picture shows a vehicle consisting of two wheels (red) and a body (dark blue) which can freely move along the horizontal direction. The wheels are made of conducting material whereas the body is an isolator. The two wheels are connected to the two poles of a light-bulb B that is fixed to the vehicle, so that an electric current can flow and the bulb can emit photons if the front and rear wheels are connected to the two poles of a battery. Such a battery providing a voltage V is kept underground with its two poles connected to conducting strips along the horizontal line along which the vehicle rolls. The ground material (light blue) is an isolator and conducting strips on its surface have a horizontal gap of length ℓ which is less than the distance L at which the wheels touch the ground if the vehicle is at rest. This means that if the vehicle is at rest and placed as in the picture, it bridges the gap between the conducting strips, the circuit will be closed and the bulb will emit photons. This remains true, at least for some time, if the vehicle slowly moves with velocity v in the horizontal direction. But what happens is v becomes large?

1. Judged from K, the rest system of the ground, the length of the vehicle and hence the distance of the wheels contracts by a factor of $\gamma^{-1} = \sqrt{1-\beta^2}$, where $\beta = v/c$. Hence, for $\gamma^{1}\ell < L$, the vehicle cannot bridge the gap anymore and no current will ever flow; there will be no photons emitted by B.

2. Judged from K', the rest system of the vehicle, the gap between the conducting strips will suffer a length contraction which makes it even easier for the vehicle to bridge the gap. Hence, for each velocity $v < c$, there is always a finite time interval during which the circuit is closed and a current can flow; there will be photons emitted.

3. Clearly, the existence of a photon is either true or false. So, who is right?

Problem 2

We consider relativistic addition for the special case of parallel velocities. If $\mathbf{v} = v\mathbf{e}_x$ is added to $\mathbf{v}_1 = v_1\mathbf{e}_x$ the result is $\mathbf{v}_2 = v_2\mathbf{e}_x$ with

$$v_2 = \frac{v + v_1}{1 + \frac{vv_1}{c^2}}\,. \tag{1}$$

1. Show that the corresponding rapidities ($\alpha := \tanh^{-1}(v)$ etc.) just add: $\alpha_2 = \alpha + \alpha_1$.

2. Apply (1) to the situation where $v_1 = c/n$ is the speed of light in the rest frame K' of a medium of refractive index n, and v is the speed of the medium relative to the frame K. Show that to leading order in $(1/c)$ the Fresnel drag-coefficient results (without involving any aether-model!):

$$v_2 \approx \frac{c}{n} + v\varphi\,, \quad \text{with} \quad \varphi = (1 - n^{-2})\,. \tag{2}$$

3. If n depends on the wavelength of the light (dispersion) we need to distinguish between the wavelength λ' measured in K' (i.e. in the rest system of the medium) und the wavelength λ measured in K. They differ due to the Doppler effect the leading order (in $1/c$) of which is just the familiar

$$\lambda' \approx \lambda\left(1 + \frac{v}{v_1}\right) = \lambda\left(1 + \frac{vn}{c}\right)\,. \tag{3}$$

   Show that this leads to the following leading-order correction to (2) if in it we re-express n, which refers to $n(\lambda')$, by $n(\lambda)$:

$$v_2 \approx \frac{c}{n} + v\varphi\,, \quad \text{with} \quad \varphi = \left(1 - n^{-2} - \frac{\lambda}{n}\frac{dn}{d\lambda}\right)\,. \tag{4}$$

Remark: The extra term $\propto dn/d\lambda$, has first been experimentally verified in 1914 by Pieter Zeeman (the same as in the “Zeeman-Effect”).

Problem 3

If we express velocities $\mathbf{v}$ in units of c (vacuum velocity of light) we call them $\boldsymbol{\beta} := \mathbf{v}/c$. In terms of β’s then the law for “adding” (denoted by a $\star$) velocities in SR, that we derived in the lecture, is

$$\boldsymbol{\beta}_2 := \boldsymbol{\beta} \star \boldsymbol{\beta}_1 = \frac{\boldsymbol{\beta} + \boldsymbol{\beta}_1^{\parallel} + \gamma^{-1}\boldsymbol{\beta}_1^{\perp}}{1 + \boldsymbol{\beta}\cdot\boldsymbol{\beta}_1}\,, \tag{5}$$

where superscript $\parallel$ and $\perp$ refer to the components parallel and perpendicular to $\boldsymbol{\beta}$, i.e. if $\mathbf{n} := \boldsymbol{\beta}/\beta$ then $\boldsymbol{\beta}_1^{\parallel} := \mathbf{n}(\mathbf{n}\cdot\boldsymbol{\beta}_1)$ and $\boldsymbol{\beta}_1^{\perp} := \boldsymbol{\beta}_1 - \boldsymbol{\beta}_1^{\parallel}$.

1. Show the two alternative forms of this law:

$$\beta_2 = \frac{\beta + \beta_1}{1 + \beta \cdot \beta_1} + \frac{\gamma}{1+\gamma}\,\frac{\beta \times (\beta \times \beta_1)}{1 + \beta \cdot \beta_1}\,, \tag{6a}$$

$$= \frac{\beta + \gamma^{-1}\beta_1}{1 + \beta \cdot \beta_1} + \frac{\gamma}{1+\gamma}\,\frac{\beta(\beta \cdot \beta_1)}{1 + \beta \cdot \beta_1}\,. \tag{6b}$$

2. Let γ, γ_1, and γ_2 denote the gamma-factors for the velocities β, β_1 and β_2, respectively. Show that

$$\gamma_2 = \gamma\gamma_1(1 + \beta \cdot \beta_1)\,. \tag{7}$$

and deduce that

$$\|\beta \star \beta_1\| = \|\beta_1 \star \beta\| < 1\,. \tag{8}$$

3. Let β and β_1 be of equal magnitude and orthogonal to each other. Calculate the magnitude of β_2 and show that the angle α between β_2 and $\beta_2' := \beta_1 \star \beta$ obeys

$$\cos(\alpha) = \frac{2\gamma}{1+\gamma^2} \tag{9}$$

Make a 2-dimensional vector-drawing showing the difference between β_2 and β_2'. This gives you an impression of the non-commutativity of $\star$.

Problem 4

We endow $\mathbb{R}^4$ with a "Minkowski metric", that is, a non-degenerate bilinear symmetric form $\eta : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ which in the standard basis $\{e_0, e_1, e_2, e_3\}$ is of diagonal form with entries of modulus 1:

$$\eta(e_a, e_b) = \mathrm{diag}(1, -1, -1, -1) \tag{10}$$

For simplicity we shall write $v \cdot w := \eta(v, w)$ and $v^2 := v \cdot v = \eta(v, v)$, just like for the euclidean inner product. Given that, we also write $\|v\|_\eta := \sqrt{|v^2|}$. Note that this is not a norm in the usual sense (why not?)

Two points $p_{1,2} \in \mathbb{R}^4$ are said to be timelike, spacelike, or lightlike separated if either $(p_1 - p_2)^2 > 0$, < 0, or $= 0$, respectively. Correspondingly, a straight line is called timelike, spacelike, or lightlike if any pair of two distinct points on it are timelike, spacelike, or lightlike separated.

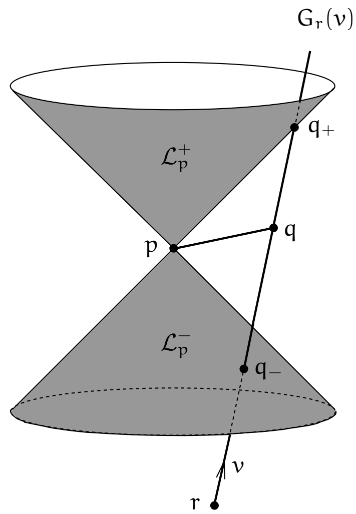
Now, let $G_r(v)$ be a timelike straight line through r in the direction of v, whose parameter representation may be given by

$$x(\lambda) = r + \lambda v\,. \tag{11}$$

Let further

$$\mathcal{L}_p := \{x \in \mathbb{R}^4 : \|x - p\|_\eta = 0\}\,. \tag{12}$$

be the lightcone at p where $p \notin G_r(V)$, with upper and lower components $\mathcal{L}_p^\pm$.



1. Prove the following reversed Cauchy-Schwarz-Theorem: For any pair of vectors v and w, where v is timelike (i.e. $v^2 > 0$) and w is arbitrary, prove the following inequality
$$(v \cdot w)^2 \geq v^2 w^2 , \tag{13}$$
with equality if and only if v and w are linearly dependent. Hint: Decompose $w = w_\parallel + w_\perp$, where $w_\parallel$ is parallel and $w_\perp$ orthogonal to v. $w_\perp$ must be spacelike and hence $w_\perp^2 \leq 0$, with equality if and only if $w_\perp = 0$.

2. Show that $G_r(v)$ intersects $\mathcal{L}_p$ in precisely two points, $q_+ \in \mathcal{L}_p^+$ and $q_- \in \mathcal{L}_p^-$.

3. Let $q \in G_r(v)$ be any point between q_+ and q_- (and different from these two). Show that
$$\|p - q\|_\eta^2 = \|q_+ - q\|_\eta \cdot \|q - q_-\|_\eta . \tag{14}$$
Hint: The vectors $(q_+ - p) = (q - p) + (q_+ - q)$ and $(q_- - p) = (q - p) + (q_- - q)$ are lightlike; hence
$$\|q - p\|_\eta^2 = (q_+ - q)^2 + 2(q - p) \cdot (q_+ - q) , \tag{15}$$
$$\|q - p\|_\eta^2 = (q_- - q)^2 + 2(q - p) \cdot (q_- - q) . \tag{16}$$
Use that $q_+ - q$ and $q - q_-$ are parallel, which means that there exists a $\lambda \in \mathbb{R}_+$ so that $q_+ - q = \lambda(q - q_-)$. Multiply (16) with λ and add that to (15).

4. Show that $(p - q)$ is η-orthogonal to $G_r(v)$, i.e. $(p - q) \cdot v = 0$, iff q is the mid-point between q_+ and q_-. Hence, all events that are Einstein synchronous with q relative to the inertial frame characterised by v are given by the 3-dimensional hyperplane through q which is η-orthogonal to v (i.e. intersecting $G_r(v)$ orthogonally).

5. Now consider two skew straight lines
$$G_r(v) := \left\{ r + \lambda v : \lambda \in \mathbb{R} \right\} , \tag{17a}$$
$$G_{r'}(v') := \left\{ r' + \lambda' v' : \lambda' \in \mathbb{R} \right\} . \tag{17b}$$

"Skew" (german: "windschief") means, that the lines do neither intersect nor are they parallel. Show that there is exactly one pair of points $(p, p') \in G_r(v) \times G_{r'}(v')$ so that p' is simultaneous to p with respect to v and p is simultaneous to p' with respect to v'.

Remark: The statement (14) is called *Robb's Theorem* [after the british geometer Alfred Arthur Robb (1873-1936)]. Note how remarkable it is by comparing its statement to the "right-triangle-altitude-theorem" (or "geometric-mean-theorem"; german: "Höhensatz") in euclidean geometry. Note in particular that in our case (14) holds for *any* point q on the $G_r(v)$!