Exercises for the lecture on

Foundations and Applications of Special Relativity

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Sheet 5

Problem 1

We denote by $\mathsf{L}((\boldsymbol{\beta}, \mathsf{D}))$ the composition of a spatial rotation $\mathsf{D} \in \mathsf{SO}(3)$ and a boost with velocity $\boldsymbol{\beta}$ (in units of c):

$$\mathsf{L}(\boldsymbol{\beta}\mathsf{D}) := \mathsf{B}(\boldsymbol{\beta})\mathsf{R}(\mathsf{D})\,. \tag{1}$$

where

$$\mathsf{B}(\boldsymbol{\beta}) := \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}^\top \\ \gamma\boldsymbol{\beta} & \mathsf{E}_3 + (\gamma-1)\mathbf{n}\otimes\mathbf{n}^\top \end{pmatrix}, \qquad \mathsf{R}(\mathsf{D}) = \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & \mathsf{D} \end{pmatrix} \tag{2}$$

1. Show
$$\mathsf{R}(\mathsf{D})\mathsf{B}(\boldsymbol{\beta})[\mathsf{R}(\mathsf{D})]^{-1} = \mathsf{B}(\mathsf{D}\boldsymbol{\beta})\,. \tag{3}$$

2. Show
$$\mathsf{L}(\boldsymbol{\beta}_1, \mathsf{D}_1)\mathsf{L}(\boldsymbol{\beta}_2, \mathsf{D}_2) = \mathsf{L}\big(\beta_1 \star \mathsf{D}_1\boldsymbol{\beta}_2\,,\, \mathsf{T}[\boldsymbol{\beta}_1, \mathsf{D}_1\boldsymbol{\beta}_2]\mathsf{D}_1\mathsf{D}_2\big) \tag{4}$$
and
$$\big[\mathsf{L}(\boldsymbol{\beta}, \mathsf{D})\big]^{-1} = \mathsf{L}(-\mathsf{D}^{-1}\boldsymbol{\beta}\,,\, \mathsf{D}^{-1}) \tag{5}$$

3. Compare this to the composition-law for the Galilei Transformations:
$$\mathsf{G}(\mathbf{v}_1, \mathsf{D}_1)\mathsf{G}(\mathbf{v}_2, \mathsf{D}_2) = \mathsf{G}\big(\mathbf{v}_1 + \mathsf{D}_1\mathbf{v}_2\,,\, \mathsf{D}_1\mathsf{D}_2\big)\,. \tag{6}$$
Answer the following questions for Lorentz and Galilei-transformations separately: a) Do pure rotations form a subgroup? b) Do pure boosts form a subgroup? c) Is the set of pure boosts invariant under conjugation.

Problem 2

Recall that the polar-decomposition of the product of two boosts is

$$\mathsf{B}(\boldsymbol{\beta}_1)\mathsf{B}(\boldsymbol{\beta}_2) = \mathsf{B}(\boldsymbol{\beta}_1 \star \boldsymbol{\beta}_2)\mathsf{R}\big(\mathsf{T}[\boldsymbol{\beta}_1, \boldsymbol{\beta}_2]\big)\,. \tag{7}$$

- Show the following properties of equivariance:

$$(\mathsf{D}\boldsymbol{\beta}_1) \star (\mathsf{D}\boldsymbol{\beta}_2) \;=\; \mathsf{D}(\boldsymbol{\beta}_1 \star \boldsymbol{\beta}_2)\,, \tag{8a}$$
$$\mathsf{T}[\mathsf{D}\boldsymbol{\beta}_1\,,\, \mathsf{D}\boldsymbol{\beta}_2] \;=\; \mathsf{D}\mathsf{T}[\boldsymbol{\beta}_1, \boldsymbol{\beta}_2]\mathsf{D}^{-1}\,. \tag{8b}$$

Problem 3

In the lecture we showed that $T[\boldsymbol{\beta}_1, \boldsymbol{\beta}_2]$ is a rotation in the oriented plane spanned by $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$. It turns $\boldsymbol{\beta}_2 \star \boldsymbol{\beta}_1$ into $\boldsymbol{\beta}_1 \star \boldsymbol{\beta}_2$ and is thus in the *negative* direction (hence clockwise) in the given orientation. The angle θ of rotation can be expressed as a function of the moduli $\beta_{1,2}$ and the (oriented) angle φ between $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ as follows:

$$\cos\theta = 1 - \frac{(\gamma_1 - 1)(\gamma_2 - 1)\,\sin^2\varphi}{1 + \gamma_1\gamma_2(1 + \beta_1\beta_2\,\cos\varphi)}\,. \tag{9}$$

1. Show that the explicit dependence on φ may be replaced by an explicit dependence on
$$\gamma := \gamma_1\gamma_2(1 + \boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2) \tag{10}$$
so that $\cos\theta$ is now a function of γ, γ_1, and γ_2, which takes the following nice symmetric form
$$\cos\theta = \frac{(1 + \gamma + \gamma_1 + \gamma_2)^2}{(1+\gamma)(1+\gamma_1)(1+\gamma_2)} - 1\,. \tag{11}$$

2. Consider the magnitudes $\beta_{1,2}$ as fixed so that $\cos(\theta)$ is a function of φ alone. This function assumes its maximal value 1 for $\varphi = 0, \pi$. Hence there must be a minimum for $\cos\theta$ at some $\varphi = \varphi_*$ in between. Show that there is precisely one such minimum at which
$$\cos\varphi_* = -\sqrt{\frac{(\gamma_1 - 1)(\gamma_2 - 1)}{(\gamma_1 + 1)(\gamma_2 + 1)}}\,. \tag{12}$$
What is the implication of the negative sign?

3. Show that the Thomas angle θ_* at φ_* obeys
$$\cos\theta_* = 1 - 2\,\frac{(\gamma_1 - 1)(\gamma_2 - 1)}{(\gamma_1 + 1)(\gamma_2 + 1)} = -\cos(2\varphi_*)\,. \tag{13}$$
Consider the special case $\beta_1 = \beta_2 = \beta$, hence $\gamma_1 = \gamma_2 = \gamma$. How large would β have to be in order for θ_* to exceed $\pi/2$ (i.e. the right angle)?