Exercises for the lecture on

Foundations and Applications of Special Relativity

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Sheet 6

Problem 1

Let V be an $n \geq 3$ dimensional real vector space with non-degenerate symmetric bilinear form $\eta : V \times V \to \mathbb{R}$ of signature $(1, n-1)$. This means, that there is a one-dimensional subspace in V restricted to which η is positive definite and an $(n-1)$-dimensional subspace restricted to which it is negative definite. The elements in V are classified as timelike, spacelike, or lightlike according to whether their square $v^2 := \eta(v, v)$ is positive, negative, or zero, respectively.

1. Define the η-orthogonal complement of a vector $w \in V$ by

$$w^{\perp} := \{v \in V : \eta(v, w) = 0\} \tag{1}$$

   and show that $w^{\perp}$ is a $(n-1)$ dimensional linear subspace that contains w iff w is lightlike, in which case η restricted to $w^{\perp}$ is degenerate.

2. Prove that if w is either timelike or spacelike, the restriction of η to $w^{\perp}$ is non-degenerate and negative-definite in the first and of signature $(1, n-2)$ in the second case.

3. Generally, we call an n'-dimensional linear subspace $V' \subset V$ timelike, spacelike or lightlike iff η restricted to V' has signature $(1, n'-1)$, $(0, n')$, or is degenerate, respectively. Apply this to the 2-dimensional plane $V' = \text{span}\{v, w\}$ and prove the following inequalities (we write $v \cdot w := \eta(v, w)$ and $v^2 := \eta(v, v)$):

$$v^2 w^2 \leq (v \cdot w)^2 \qquad \text{if span}\{v, w\} \text{ is timelike}\,, \tag{2a}$$

$$v^2 w^2 \geq (v \cdot w)^2 \qquad \text{if span}\{v, w\} \text{ is spacelike}\,, \tag{2b}$$

$$v^2 w^2 = (v \cdot w)^2 \qquad \text{if span}\{v, w\} \text{ is lightlike}\,. \tag{2c}$$

   These triple of equations replace the single Cauchy-Schwarz inequality for non-positive-definite inner products.

Problem 2

Let V be a $n \geq 3$ real vector space with non-degenerate symmetric bilinear form $\eta : V \times V \to \mathbb{R}$. Let $f : V \to V$ be a map that preserves the inner product; i.e. $\eta\big(f(v)\,,\, f(w)\big) = \eta(v, w)$ for all $v, w \in V$.

- Prove that if f is surjective it must be linear and hence an isomorphism.

Hint: Consider $I := \eta\big(af(u) + bf(v) - f(au + bv)\,,\, w\big)$, where $a, b \in \mathbb{R}$ and $u, v, w \in V$. Use the properties of f and η to show that $I = 0$.