Exercises for the lecture on

Foundations and Applications of Special Relativity

von DOMENICO GIULINI

Sheet 7

Problem 1

Let N and Q be groups and $\alpha : Q \to \mathrm{Aut}(N)$ a homomorphism. The semi-direct product $N \rtimes_\alpha Q$ of N with Q relative to α is defined by the following multiplication map on the set $N \times Q$ (compare last lecture):

$$(n_1, q_1)(n_2, q_2) = (n_1 \alpha_{q_1}(n_2)\,,\; q_1 q_2)\,. \tag{1}$$

1. Show that (e_N, e_Q) is the group identity, that the inverse of (n, q) is $(n, q)^{-1} = \big(\alpha_q^{-1}(n^{-1})\,,\; q^{-1}\big)$, and that the group multiplication (1) is associative. Here e_N and e_Q denote the identities in N and Q, respectively.

2. Consider the following subgroups in $N \rtimes_\alpha Q$:
$$\begin{aligned} N' &:= N \times \{e_Q\} = \big\{(n, e_Q) : n \in N\big\}\,, \\ Q' &:= \{e_N\} \times G\} = \big\{(e_N, q) : q \in Q\big\}\,, \end{aligned} \tag{2}$$
and show that N' is always normal, whereas Q' is normal iff α is trivial (i.e. $\mathrm{kernel}(\alpha) = Q$).

3. Let $Q'_n := (n, e_Q)Q'(n, e_Q)^{-1}$ (conjugation of Q' with an element $(n, e_Q) \in N'$). What elements are in the intersection $Q'_{n_1} \cap Q'_{n_2}$ for n_1 and n_2 different elements in N? Specialise this to the case $\mathbb{R}^3 \rtimes SO(3)$ and give an interpretation of this intersection.

4. Prove that a group G is the semi-direct product $N \rtimes_\alpha Q$ for some α iff it contains two subgroups N' and Q' isomorphic to N and Q, respectively, such that 1) N' is normal, 2) $N'Q' := \{n'q' : n' \in N',\; q' \in Q'\} = G$, and 3) $N' \cap Q' = e_G$.

   Hint: Consider the embeddings (injective homomorphisms) $i : N \to G$ and $j : Q \to G$ the images of which are N' and Q', respectively. Consider the map $\phi : N \times Q \to G$, $\phi(n, q) = i(n)j(q)$ and show that it is a group isomorphism if we endow $N \times Q$ with the semi-direct product structure relative to $\alpha : Q \to \mathrm{Aut}(N)$, where
$$\alpha_q(n) := i^{-1}_{N'}\big[j(q)i(n)j(q^{-1})\big]\,. \tag{3}$$
   and $i^{-1}_{N'}$ denotes the inverse of i restricted to N'.

5. Another characterisation of a semi-direct product is this: Let G be a group admitting a projection homomorphism $\pi : G \to Q$ onto a subgroup $Q \subset G$. (Note: The word "projection" entails that the map π is surjective and idempotent; i.e. $\pi \circ \pi = \pi$). Then G is a semi-direct product of the kernel N and the image Q of π. Prove that statement.

Problem 2

Consider $(1+2)$-dimensional Minkowski space represented in an affine chart by $\mathbb{R}^3$ with the metric $\eta_{ab} = \text{diag}(1,-1,-1,)$. The coordinates are called $(x^0, x^1, x^2) = (ct, x, y)$. Consider the circle $x^2+y^2 = R^2$ in space which we think of as an (infinitely thin) fibre-glass cable as wave-guide, inside which light can travel tangentially. Assume that at time $t = 0$ two photons of equal frequency are emitted from a point on the circle, say $(x = 0, y = -R)$, one in each direction of the wave-guide.

Assume that simultaneous and equilocal to the photon-emission event, i.e. at $(t = 0, x = 0, y = -R)$, an observer starts running along the circular wave-guide, say in a counterclockwise direction, with constant angular velocity Ω, where $\Omega R < c$. On his/her way the observer will first meet the clockwise, then the counterclockwise circling photon.

1. Calculate the observer's proper time difference between these two meeting events as a function of Ω.

   Hint: The proper time of the observer is the length of the observer's world-line divided by c measured in the metric η. The three worldlines (one for each photon, one for the observer) lie in the 2-dimensional cylinder $x^2 + y^2 = R^2$ in 3-dimensional Minkowski spacetime. Think of this cylinder as being cut open along the generator $x = 0$, $y = R$ parallel to the t-axis and unfold it to a timelike planar strip of widths $2\pi R$ in the $ct - x$ plane . In this fashion the problem reduces to a simple geometric problem in $(1+1)$-dimensional Minkowski spacetime.

2. What would happen if one tried to synchronise the clocks along the waveguide in such a way that with respect to them the moving observer measures the same velocity c of light in both directions?

   Hint: Take the picture of the planar strip and draw into it the line of Einstein-simultaneity for the observer.

3. What has all this to do with the Sagnac-Effect?