Exercises for the lecture on

Foundations and Applications of Special Relativity

von DOMENICO GIULINI

Sheet 8

Problem 1

The so-called four-momentum of a point particle moving on a timelike trajectory in Minkowski space is given by $p = mv$, where $v = cu$ and $u \in V$ with $\eta(u, u) = 1$ is a state of motion and m is a constant called the rest-mass of the particle. If the "particle" has zero rest mass (photon) p is lightlike. If $\{e_0, e_1, e_2, e_3\}$ is an orthonormal basis of (V, η) with dual basis $\{\theta^0, \theta^1, \theta^2, \theta^3\}$ so that $\eta(e_a, e_b) = \mathrm{diag}(1, -1, -1, -1)$, we call $p^0 := \theta^0(p) = E/c$, where E is the "energy" of the particle with respect to the reference frame represented by $\{e_0, e_1, e_2, e_3\}$, and $p^a = \theta^a(p)$ $(a = 1, 2, 3)$ the a-th component of the momentum.

Consider the case of the elastic collision of two particles, the first with rest mass zero and the second with rest mass $m > 0$. The assumption of elasticity here means that the rest-masses before and after collision are the same for each particle. Think of the first particle as being a photon with four momentum p_1, so its energy in the specified frame is $\hbar\omega$, with ω the photon's angular frequency in that frame, and $p^a = \hbar k^a$, with $\mathbf{k} = (k^1, k^2, k^3)$ the components of the wave-vector. That p is lightlike is equivalent to the standard dispersion relation $\omega^2 = c^2\|\mathbf{k}\|^2$.

Assume the massive particle to be at rest initially and consider energy-momentum conservation in the given frame:

$$p_1 + p_2 = p_1' + p_2' \,. \tag{1}$$

where the primed quantities are those after collision. Prove Compton's formula, according to which the wavelength $\lambda' = 2\pi c/\omega'$ of the photon after scattering is related to the wavelength $\lambda = 2\pi c/\omega$ before scattering by

$$\lambda' - \lambda = \frac{h}{mc}(1 - \cos\varphi)\,, \tag{2}$$

where φ is the scattering angle and $h = 2\pi\hbar$. Hint: Note that $p_1^2 = p'^2_1 = 0$ and that $p_2^2 = p'^2_2 = mc^2$ (elasticity assumption). Use (1) to show $p_1 \cdot p_2 = p_1' \cdot p_2'$. Next multiply (1) with p_1' and derive $p_1 \cdot p_1' + p_2 \cdot p_1' = p_1 \cdot p_2$ and from that (2).

Problem 2

Consider a real $n \geq 3$ vector space V with non-degenerate symmetric bilinear form η of signature $(1, n-1)$ (i.e. Minkowski metric). Via η we identify V^* with V, which allows us to identify $\mathrm{End}(V) \simeq V \otimes V^*$ with $V \otimes V$. An element $u \otimes v \in V \otimes V$ then acts as a linear map on $w \in V$ via

$$(u \otimes v)w = (v \cdot w)\, u \tag{3}$$

where we also used the abbreviation

$$v \cdot w := \eta(v, w)\,, \tag{4}$$

The composition of the two linear maps denoted by $(u \otimes v)$ and $(u' \otimes v')$ is then given by

$$(u \otimes v) \circ (u' \otimes v') = (v \cdot u')\, u \otimes v'\,. \tag{5}$$

In this way we know how to form powers and sums of powers of elements of the form $u \otimes v$, which just form the associative algebra $\mathrm{End}(V)$.

1. We write
$$u \wedge n := u \otimes n - n \otimes u\,. \tag{6}$$
Show that for any pair (u, n) of vectors the map $u \wedge n \in \mathrm{End}(V)$ is anti-self-adjoint with respect to η.

2. Show that the map
$$L(u, n) := \exp(u \wedge n) := \sum_{k=0}^{\infty} \frac{(u \wedge n)^k}{k!} \tag{7}$$
is a proper orthochronous Lorentz transformation.

3. Specialise to the case where u and n are orthogonal with u timelike and v spacelike:
$$(u \cdot n) = 0\,, \quad u^2 = -n^2 = 1\,, \tag{8}$$
Prove that
$$(u \wedge n)^2 := (u \wedge n) \circ (u \wedge n) = P_{(u,n)} \tag{9}$$
where $P_{(u,n)}$ is the orthogonal projection (with respect to η) onto the timelike 2-dimensional subspace (2-plane) span$\{u, n\} \subset V$. Prove further that
$$B(u, n, \rho) := \exp(-\rho u \wedge n) = \mathrm{id}_V + \big(\cosh(\rho) - 1\big) P_{(u,n)} - \sinh(\rho)\, u \wedge n\,, \tag{10}$$
and that this is an active boost-transformation with rapidity $\rho = \tanh^{-1}(\beta)$ in the plane span$\{u, v\}$. Split the exponential series into its odd and even powers, use (8), and take care with the zeroth-order term, which is just id_V. After proving (9), apply the result to u, n, and any vector orthogonal to span$\{u, v\}$ and see what you get.

4. Suppose (u', n') is another pair of vectors that satisfies (8), span$\{u', v'\}$ = span$\{u, v\}$, and that endows the 2-plane with the same orienation. Show that $B(u, v, \rho) = B(u', v', \rho)$. Hint: You can prove this in words, without any calculation. Can you write the same transformation by choosing u and n both lightlike?

5. Repeat the same calculation, now with u and v both spacelike and orthonormal. Show that you just get (10), but now with cos and sin replacing cosh and sinh. Show that this gives a rotation in the spacelike (oriented) 2-plane span$\{u, v\}$ by an angle $(-\rho)$.

6. Repeat the calculation once more, now with u lightlike and n spacelike, normalised, and othogonal to u. Hint: In this case the exponential series is finite.