Exercises for the lecture on

Foundations and Applications of Special Relativity

von DOMENICO GIULINI

Sheet 9

Problem 1

Consider a process $A + B \to C$ where two particles A and B annihilate each other and create a third particle C. The corresponding 4-momenta are p_A, p_B and p_C, respectively. Show that p_C cannot be lightlike if at least one of p_A and p_B are timelike. Hint: Apply the appropriate Cauchy-Schwarz inequality.

Problem 2

Consider two particles with masses m_1 and m_2 and 4-momenta p_1 and p_2, respectively. We have $p_a^2 := \eta(p_a, p_a) = m_a^2 c^2$. The 4-momentum P of the combined system is given by the sum $p_1 + p_2$.

Generally, if u is a unit timelike vector representing an inertial system, the energy of a system with 4-momentum p in that system is $E_u := cp \cdot u$. Apply this to the 2-particle system and calculate the energy E_{cm} in the "centre-of-mass" system, where u is proportional to $p_1 + p_2$, and also the energy E_ℓ in the "laboratory system", where one of system, say the second, is at rest, i.e. u is proportional to p_2. Express E_{cm} as a function E_ℓ. Observe how E_{cm} grows as a function of E_ℓ for increasingly larger E_ℓ? What conclusion do you draw from this?

Hint: The calculation will give you E_{cm} as well as E_ℓ as function of $(p_1 \cdot p_2) := \eta(p_1, p_2)$, m_1, and m_2. This allows you to eliminate $p_1 \cdot p_2$ and express E_{cm} through E_ℓ, m_1, and m_2.

Problem 3

A particle P_0 of mass m_0 decays into two particles P_1 and P_2 of masses m_1 and m_2, respectively. Show that the energy and modulus of 3-momentum $\|\mathbf{p}_1\|$ of P_1 in the rest frame of P_0 are

$$E_1 = c^2 \frac{m_0^2 + m_1^2 - m_2^2}{2m_0}, \tag{1a}$$

$$\|\mathbf{p}_1\| = c \frac{\sqrt{\left[m_0^2 - (m_1 + m_2)^2\right]\left[m_0^2 - (m_1 - m_2)^2\right]}}{2m_0}, \tag{1b}$$

and with indices 1 and 2 exchanged for particle P_2. Note that the right-hand of side of (1a) is non-symmetric under $1 \leftrightarrow 2$ whereas (1b) is symmetric. Why is that?

What are the magnitudes of the velocities of P_1 and P_2 in the rest frame of P_0?

Problem 4

This exercise is an extension of the previous Problem 3. Show that the magnitude of the relative velocity β_{10} (in units of c) of P_1 in the rest frame of P_0 is

$$\beta_{10} = \frac{\sqrt{\left[m_0^2 - (m_1 + m_2)^2\right]\left[m_0^2 - (m_1 - m_2)^2\right]}}{m_0^2 + m_1^2 - m_2^2} \tag{2}$$

and similarly for β_{20} with indices exchanged, and that the relative velocity β_{12} (in units of c) of P_1 in the rest frame of P_2 is

$$\beta_{12} = \frac{\sqrt{\left[m_0^2 - (m_1 + m_2)^2\right]\left[m_0^2 - (m_1 - m_2)^2\right]}}{m_0^2 - m_1^2 - m_2^2} \tag{3}$$

which equals β_{21} Hint: The first equation follows from (1). For the second we recall that the relative velocity between two states of motion (timelike normalised vectors) u_1 and u_2 is the velocity of the boost in the plane $\mathrm{span}\{u_1, u_2\}$ that transforms one into the other. Now, $u_1 \cdot u_2 = \gamma = 1/\sqrt{1-\beta^2}$. In our case $u_1 = p_1/(m_1 c)$, $u_2 = p_2/m_2 c$, and $(m_0 c)^2 = (p_1 + p_2)^2 = (m_1 c)^2 + (m_2)^2 + 2(p_1 \cdot p_2)$.

Problem 5

A particle of charge e moves in a constant electric field $\mathbf{E} = E\mathbf{n}$. Here E is constant and $\mathbf{n}$ is also constant and normalised $\|n\| = 1$. The Lagrangian is

$$L(x, \dot{\mathbf{x}}) = -mc^2\sqrt{1 - \frac{\dot{\mathbf{x}}^2}{c^2}} + eE\,(\mathbf{x} \cdot \mathbf{n})\,. \tag{4}$$

Solve the equations of motion with initial conditions $\mathbf{x}(t = 0) = \dot{\mathbf{x}}(t = 0) = \mathbf{0}$. Describe the graph of the function $t \mapsto \mathbf{x}(t)$ geometrically.