Sheet 1 : Solutions

Problem 1

$$\ddot{\vec{x}}_a = -G \sum_{\substack{b=1 \\ b\neq a}}^{n} m_b \frac{\vec{x}_a(t)-\vec{x}_b(t)}{\|\vec{x}_a(t)-\vec{x}_b(t)\|^3} \quad (1.1.1)$$

The Galilei group consists of the following transformations

$$T_b : (t,\vec{x}) \mapsto (t',\vec{x}') := (t+b, \vec{x})$$

(time translations) (1.1.2a)

$$T_{\vec{a}} : (t,\vec{x}) \mapsto (t',\vec{x}') := (t, \vec{x}+\vec{a})$$

(space translations) (1.1.2b)

$$B_{\vec{v}} : (t,\vec{x}) \mapsto (t',\vec{x}') := (t, \vec{x}+\vec{v}\,t)$$

(boosts ≡ velocity transf.) (1.1.2c)

$$R_D : (t,\vec{x}) \mapsto (t',\vec{x}') := (t, D\vec{x})$$

(spatial rotations) (1.1.2d)

Here $b \in \mathbb{R}$, $\vec{a}, \vec{v} \in \mathbb{R}^3$ (1.1.2e)

$$D \in SO(3) := \{ A \in GL(\mathbb{R}^3) : A^T = A^{-1} \text{ and } \det(A) = 1 \} \quad (1.1.2f)$$

The general Galilei transformation is a combination of all these:

$$g(b, \vec{a}, \vec{v}, D) := T_b \circ T_{\vec{a}} \circ B_{\vec{v}} \circ R_D \quad (1.1.3)$$

Hence $g(b, \vec{a}, \vec{v}, D)\,(t, \vec{x})$

$$= (t+b,\ \vec{a} + \vec{v}t + D\vec{x}) \quad (1.1.4)$$

The action of $g(b, \vec{a}, \vec{v}, D)$ on a function

$$\vec{x} : \mathbb{R} \to \mathbb{R}^3$$
$$t \mapsto \vec{x}(t) \quad (1.1.5)$$

is as follows

$$T_g \vec{x}(t) = \vec{x}\,'(t)$$
$$= \vec{a} + \vec{v}t + D\vec{x}(t-b) \quad (1.1.6)$$

We will see this in full generality and mathematical detail in the lecture. Note that $\vec{x}\,'(t) = \vec{x}(t-b)$ shifts the graph of $\vec{x}$ by an amount b in positive t-direction.

(1.1.6) now applies to every $\vec{x}_a$:

$$T_g \vec{x}_a(t) = \vec{x}\,'_a(t) = \vec{a} + \vec{v}t + D\vec{x}_a(t-b)$$

Hence

$$\vec{x}'_a(t) - \vec{x}'_b(t) = D\left(\vec{x}_a(t-b) - \vec{x}_b(t-b)\right)$$

$$\|\vec{x}'_a(t) - \vec{x}'_b(t)\| = \|\vec{x}_a(t-b) - \vec{x}_b(t-b)\|$$

$$\frac{d^2}{dt^2}\vec{x}'_a(t) = D\frac{d^2}{dt^2}\vec{x}_a(t-b)$$

or $$\ddot{\vec{x}}'_a(t) = D\ddot{\vec{x}}_a(t-b).$$

Therefore

$$\ddot{\vec{x}}'_a(t) = -G\sum_{\substack{b=1\\ b\neq a}}^{n} m_b \frac{\vec{x}'_a(t) - \vec{x}'_b(t)}{\|\vec{x}'_a(t) - \vec{x}'_b(t)\|^3}$$

is equivalent to

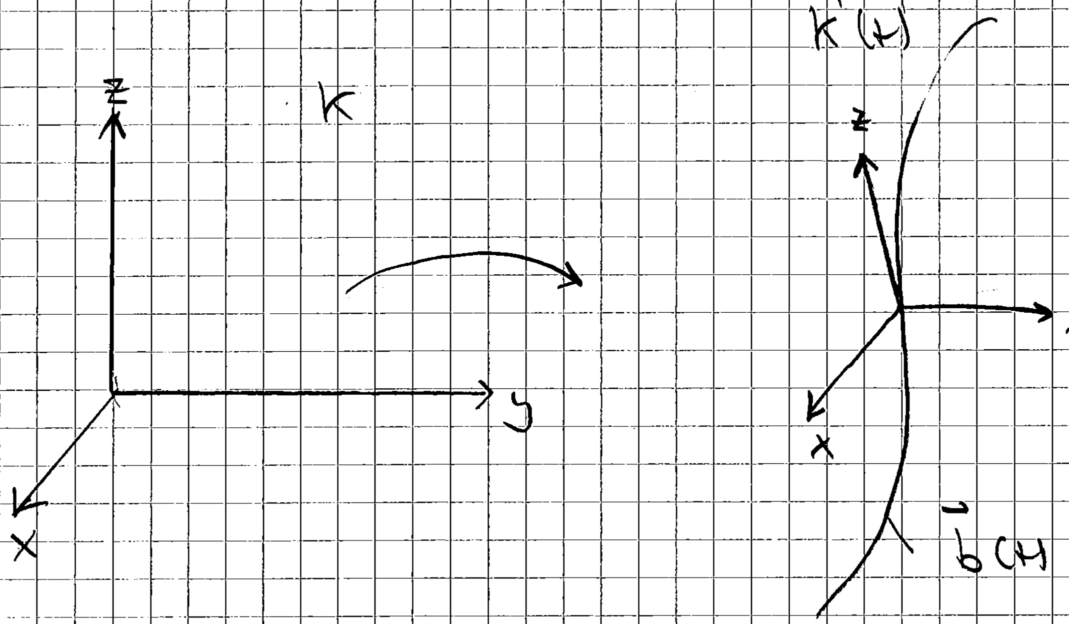
$$D\ddot{\vec{x}}_a(t-b) = -G\sum_{\substack{b=1\\ b\neq a}}^{n} m_b\, D\left(\frac{\vec{x}_a(t-b) - \vec{x}_b(t-b)}{\|\vec{x}_a(t-b) - \vec{x}_b(t-b)\|^3}\right)$$

$$\Leftrightarrow \ddot{\vec{x}}_a(t) = -G\sum_{\substack{b=1\\ b\neq a}}^{n} m_b \frac{\vec{x}_a(t) - \vec{x}_b(t)}{\|\vec{x}_a(t) - \vec{x}_b(t)\|^3}$$

Since D is invertible and equations holds for all t.

Notice that we have shown: the functions $\vec{x}'_a(t)$ satisfy (1.1.1) if and only if $\vec{x}_a(t)$ do. This means that the Galilei group acts as symmetries of (1.1.1).

Problem 2



(1.2.1)

The right frame moves with its origin on the curve $t \longmapsto \vec b(t)$ as seen from K. The axes are rotated against that of K by $D(t) \in SO(3)$.

A given space curve can be described w.r.t. to K by $\vec X(t)$ and w.r.t. K' by $\vec X'(t)$.

$$\vec X(t) = X^a(t)\,\vec e_a$$
$$= \vec b(t) + X'^a(t)\,\vec e\,'_a(t) \qquad (1.2.2)$$

The relation between the basis $\{\vec e_a,\ a=1,2,3\}$ in K and the basis $\{\vec e\,'_a(t),\ a=1,2,3\}$ in $K'(t)$ is

$$\vec e\,'_a(t) = D^b{}_a(t)\,\vec e_b. \qquad (1.2.3)$$

Hence

$$X^a(t)\,\vec{e}_a = b^a(t)\,e_a + D^a{}_b(t)\,X'^b(t)\,\vec{e}_a$$

or

$$X^a(t) = b^a(t) + D^a{}_b(t)\,X'^b(t) \qquad (1.2.4)$$

Note that $\vec{X} = (X^1, X^2, X^3)$ and $\vec{b} = (b^1, b^2, b^3)$ refer to $\{e_a, a=1,2,3\}$, whereas $\vec{X}' = (X'^1, X'^2, X'^3)$ refer to $\{\vec{e}'_a(t), a=1,2,3\}$

The Newtonian equation of motion for a point-particle of mass m with respect to inertial frame K is

$$\vec{F}/m = \ddot{\vec{X}}(t) \qquad (1.2.5)$$

where $\vec{F} = (F^1, F^2, F^3)$ are the components of the force with respect to K. The components with respect to K' are obtained from

$$F^a\,\vec{e}_a = F'^b\,\vec{e}'_b = F'^b\,D^a{}_b(t)\,e_a$$

$$\rightarrow \quad F^a = D^a{}_b(t)\,F'^b \qquad (1.2.6)$$

or $$F'^a = [D^{-1}]^a{}_b(t)\,F^b \qquad (1.2.7)$$

In order to evaluate Newton's equation (1.25) in terms of components with respect to non-inertial frames, we have to understand the notion of angular velocity.

From

$$\vec{e}'_a(t) = D^b{}_a(t)\, \vec{e}_b$$

we get by time-differentiation

$$\dot{\vec{e}}'_a(t) = \dot{D}^b{}_a(t)\, \vec{e}_b$$

$$= \dot{D}^b{}_a(t) \left(D^{-1}(t)\right)^c{}_b\, \vec{e}'_c(t)$$

$$= (D^{-1}\dot{D})^c{}_a\, \vec{e}'_c$$

$$=: \Omega'^c{}_a\, \vec{e}'_c$$

where

$$\Omega'^c{}_a(t) := \left(D^{-1}(t)\,\dot{D}(t)\right)^c{}_a$$

$$= \left[D^{-1}(t)\right]^c{}_b\, \dot{D}^b{}_a(t)$$

(summation over repeated index b being understood; → Einstein's summation convention)

The coefficients $\Omega'^c{}_a(t)$ are the components of the <u>angular-momentum endomorphism</u> with respect to the frame K'. The components of the very same map with respect to the frame K are

$$\Omega^c{}_a(t) = D^c{}_n(t)\,\Omega'^n{}_m(t)\,[D^{-1}]^m{}_a(t)$$
$$= \dot{D}^c{}_b(t)\,[D^{-1}(t)]^b{}_a$$

In analytical mechanics one calls

$$\Omega'^c{}_a = [D^{-1}]^c{}_b\,\dot{D}^b{}_a$$

and $\Omega^c{}_a = \dot{D}^c{}_b\,[D^{-1}]^b{}_a$

the angular velocity endomorphism components with respect to the "body-fixed" and "space-fixed" frame, respectively. If

$$e'_a = D^b{}_a\,e_b$$

relates the basis vectors and

$$\theta'^a = [D^{-1}]^a{}_b\,\theta^b$$

the corresponding dual vectors, so that

$$\theta^a(e_b) = \theta'^a(e'_b) = \delta^a{}_b$$

then

$$\Omega = \Omega^c{}_a\,e_c \otimes \theta^a = \Omega'^c{}_a\,e'_c \otimes \theta'^a$$

The angular-momentum endomorphism is antisymmetric with respect to the euclidean inner product

$$\langle e_a, e_b \rangle = \langle e'_a, e'_b \rangle = \delta_{ab}$$

i.e.

$$\langle V, \Omega W \rangle = - \langle \Omega V, W \rangle$$

$\forall V, W$. In components that means e.g. for e_a,

$$\langle e_a, \Omega e_b \rangle = \langle e_a, \Omega^c{}_b e_c \rangle$$

$$= \delta_{ac} \Omega^c{}_b =: \Omega_{ab}$$

$$= - \langle \Omega e_a, e_b \rangle$$

$$= - \langle \Omega^c{}_a e_c, e_b \rangle$$

$$= - \delta_{bc} \Omega^c{}_a = - \Omega_{ba}$$

or

$$\Omega_{ab} = - \Omega_{ba}$$

where $\Omega_{ab} := \delta_{ac} \Omega^c{}_b$

The same is true for the comp. Ω'_{ab}:

$$\Omega'_{ab} = - \Omega'_{ba}$$

since e'_a is also orthonormal.

In 3-dimensions, any antisymmetric endomorphism Ω is of the form

$$\Omega \vec{v} = \vec{W} \times \vec{V}$$

In fact, if we define

$$W^a = -\frac{1}{2} \varepsilon^{abc} \Omega_{bc} \qquad \Big/ \varepsilon_{anm}$$

$$W^a \varepsilon_{anm} = -\frac{1}{2} \varepsilon^{abc} \varepsilon_{anm} \Omega_{bc}$$

$$= -\frac{1}{2} (\delta^b_n \delta^c_m - \delta^b_m \delta^c_n) \Omega_{bc}$$

$$= - \Omega_{nm}$$

i.e.

$$W^a = -\frac{1}{2} \varepsilon^{abc} \Omega_{bc}$$

inverse $\Omega_{ab} = - W^c \varepsilon_{cab}$

$$= - \varepsilon_{abc} W^c$$

then

$$\Omega^a{}_b X^b = - \varepsilon^a{}_{bc} W^c X^b$$

$$= + \varepsilon^a{}_{cb} W^c X^b$$

$$= (\vec{W} \times \vec{X})^a$$

In the same fashion have

$$\omega'^{a} = -\frac{1}{2}\,\varepsilon'^{abc}\,\Omega'_{bc}$$

$$\Omega'_{ab} = -\,\varepsilon'_{abc}\,\omega'^{c}$$

with angular momentum <u>vector</u>

$$\omega = \omega^{a}\,e_a = \omega'^{b}\,e'_b$$

$$= \omega'^{b}\,D^{a}{}_{b}\,e_a$$

$$\Leftrightarrow \quad \omega'^{a} = [D^{-1}]^{a}{}_{b}\,\omega^{b}$$

This indeed follows from above:

$$\Omega = \Omega^{a}{}_{b}\,e_a \otimes \theta^{b} = \Omega'^{a}{}_{b}\,e'_a \otimes \theta'^{b}$$

$$= \Omega'^{a}{}_{b}\,D^{n}{}_{a}\,[D^{-1}]^{b}{}_{m}\,e_n \otimes \theta^{m}$$

$$\Rightarrow \Omega^{n}{}_{m} = D^{n}{}_{a}\,\Omega'^{a}{}_{b}\,[D^{-1}]^{b}{}_{m}$$

$$\Leftrightarrow \Omega_{nm} = \Omega'_{ab}\,[D^{-1}]^{a}{}_{n}\,[D^{-1}]^{b}{}_{m}$$

$$\Leftrightarrow \Omega'_{ab} = \Omega_{nm}\,D^{n}{}_{a}\,D^{m}{}_{b}$$

$$\Rightarrow \quad \omega'^{a} = -\frac{1}{2}\,\varepsilon'^{abc}\,\Omega'_{bc}$$

$$= -\frac{1}{2}\,\varepsilon'^{abc}\,D^{n}{}_{b}\,D^{m}{}_{c}\,\Omega_{nm}$$

Have

$$\varepsilon'^{lnm} = D^l{}_k D^n{}_b D^m{}_c \varepsilon'^{kbc}$$

multiplication with $[D^{-1}]^a{}_l$

$$\Rightarrow [D^{-1}]^a{}_l \varepsilon^{lnm} = D^n{}_b D^m{}_c \varepsilon'^{abc}$$

$$\Rightarrow \omega'^a = -\tfrac{1}{2}[D^{-1}]^a{}_l \varepsilon^{lnm} \Omega_{nm}$$

$$= [D^{-1}]^a{}_l \omega^l \qquad \square$$

ω^a = angular momentum components in "space-fixed" frame

ω'^a = " in "body-fixed" frame

Back to Newton's equation:

$$\vec{X}(t) = \vec{b}(t) + D(t)\vec{X}'(t')$$

$$\dot{\vec{X}} = \dot{\vec{b}} + \dot{D}\vec{X}' + D\dot{\vec{X}}'$$

$$= \dot{\vec{b}} + D(D^{-1}\dot{D}\vec{X}' + \dot{\vec{X}}')$$

$$= \dot{\vec{b}} + D(\vec{\omega}' \times \vec{X}' + \dot{\vec{X}}')$$

$$\ddot{\vec{X}} = \ddot{\vec{b}} + \dot{D}\left(\vec{\omega}' \times \vec{X}' + \dot{\vec{X}}'\right) + D\left(\dot{\vec{\omega}}' \times \vec{X}' + \vec{\omega}' \times \dot{\vec{X}}' + \ddot{\vec{X}}'\right)$$

$$= \ddot{\vec{b}} + D\left[D^{-1}\dot{D}\left(\vec{\omega}' \times \vec{X}' + \dot{\vec{X}}'\right) + \dot{\vec{\omega}}' \times \vec{X}' + \vec{\omega}' \times \dot{\vec{X}}' + \ddot{\vec{X}}'\right]$$

$$= \ddot{\vec{b}} + D\left[\vec{\omega}' \times \left(\vec{\omega}' \times \vec{X}' + \dot{\vec{X}}'\right) + \dot{\vec{\omega}}' \times \vec{X}' + \vec{\omega}' \times \dot{\vec{X}}' + \ddot{\vec{X}}'\right]$$

$$= \ddot{\vec{b}} + D\left[\vec{\omega}' \times (\vec{\omega}' \times \vec{X}') + 2\,\vec{\omega}' \times \dot{\vec{X}}' + \dot{\vec{\omega}}' \times \vec{X}' + \ddot{\vec{X}}'\right]$$

The left-hand side is $\vec{F}/m$

Multiplying the whole equation by D^{-1} and setting

$D^{-1}\ddot{\vec{b}} = \vec{a}' =$ acceleration comp. K' against K in $\vec{e}'_a$-basis

$D^{-1}\vec{F} = \vec{F}' =$ force comp. in $\vec{e}_a$-basis

Solving for $\ddot{\vec{X}}'$ gives

$$\ddot{\vec{X}}' = \vec{F}' \qquad \text{force}$$
$$- \vec{a}' \qquad \text{lin. acc.}$$
$$- \vec{\omega}' \times (\vec{\omega}' \times \vec{X}') \qquad \text{centrifugal}$$
$$- 2\,\vec{\omega}' \times \dot{\vec{X}}' \qquad \text{coriolis}$$
$$- \dot{\vec{\omega}}' \times \vec{X}' \qquad \text{Euler term}$$

Problem 3

$$M := \{ (X^0, X^1, X^2, X^3)^T \in \mathbb{R}^4 : X^0 = 1 \}$$

is a hyperplane in $\mathbb{R}^4$ not containing the origin, i.e. M is <u>not</u> a vector-subspace. Rather, it is an affine sub-space of dimension 3 over the vector space $V = \{ w \in \mathbb{R}^4 : \langle e_0, w \rangle = 0 \}$, i.e V = orthogonal complement to $e_0 = (1, 0, 0, 0)^T$.

Euclidean motions in $\mathbb{R}^3$ are given by

$$\vec{X} \mapsto \vec{X}' = \underset{\substack{\uparrow \\ \text{Translations}}}{\vec{a}} + \underset{\substack{\uparrow \\ \text{Rotations}}}{D}\vec{X}$$

On the other hand

$$g(\vec{a}, D) \begin{pmatrix} 1 \\ \vec{X} \end{pmatrix} = \begin{pmatrix} 1 & \vec{0}^T \\ \vec{a} & D \end{pmatrix} \begin{pmatrix} 1 \\ \vec{X} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ \vec{a} + D\vec{X} \end{pmatrix}$$

Next

$$\begin{pmatrix} 1 & \vec{0}^T \\ \vec{a}_1 & D_1 \end{pmatrix} \begin{pmatrix} 1 & \vec{0}^T \\ \vec{a}_2 & D_2 \end{pmatrix} = \begin{pmatrix} 1 & , \vec{0}^T \\ \vec{a}_1 + D_1 \vec{a}_2, & D_1 D_2 \end{pmatrix}$$

$$\rightarrow \; g(\vec{a}_1, D_1)\, g(\vec{a}_2; D_2) = g(\vec{a}_1 + D_1 \vec{a}_2, D_1 D_2)$$

This puts a group structure on the set

$$\mathbb{R}^3 \times SO(3)$$

which is called a semi-direct product. The neutral / identity element is given by $e = g(\vec{0}, E_3)$ and the inverse of $g(\vec{a}, \bar{D})$ is

$$[g(\vec{a}, \bar{D})]^{-1} = g(-\bar{D}^{-1}\vec{a}, \bar{D}^{-1})$$

This we check:

$$g(-\bar{D}^{-1}\vec{a}, \bar{D}^{-1})\, g(\vec{a}, D)$$

$$= g(-\bar{D}^{-1}\vec{a} + \bar{D}^{-1}\vec{a}, \bar{D}^{-1}D) = g(\vec{0}, E_3)$$

or

$$g(\vec{a}, D)\;\; g(-\bar{D}^{-1}\vec{a}, \bar{D}^{-1})$$

$$= g(\vec{a} - D\bar{D}^{-1}\vec{a}, D\bar{D}^{-1}) = g(\vec{0}, E_3)$$

Associativity may also be checked directly, but also follows from the associativity of matrix multiplication.

We consider the subgroups

$$T := \{g(\vec{a}, D) : D = E_3\}$$

$$R := \{g(\vec{a}, \vec{D}) : \vec{a} = \vec{0}\}$$

both contain the identity element $g(\vec{0}, E_3)$ and are closed under multiplication :

$$g(\vec{a}_1, E_3)\, g(a_2, E_3) = g(\vec{a}_1 + \vec{a}_2, E_3)$$

$$g(\vec{0}, D_1)\, g(\vec{0}, D_2) = g(\vec{0}, D_1 D_2)$$

T is invariant :

$$g(\vec{a}, D)\; g(\vec{a}', E_3)\,[g(\vec{a}, D)]^{-1}$$

$$= g(\vec{a} + D\vec{a}', D)\; g(-D^{-1}\vec{a}, D^{-1})$$

$$= g(\vec{a} + D\vec{a}' - \vec{a}, E_3)$$

$$= g(D\vec{a}', E_3)$$

= pure translation with vector $D\vec{a}'$

R is <u>not</u> invariant :

$$g(\vec{a}, D)\; g(\vec{0}, D')\,[g(\vec{a}, D)]^{-1}$$

$$= g(\vec{a}, DD')\; g(-D^{-1}\vec{a}, D^{-1})$$

$$= g(\vec{a} - DD'D^{-1}\vec{a}, DD'D^{-1})$$

This is in R iff $\vec{a} - DD'D^{-1}\vec{a} = \vec{0}$

which is equivalent to

$$D^{-1}\vec{a} = D' D^{-1}\vec{a}$$

i.e. $D^{-1}\vec{a}$ is an eigenvector of D' with eigenvalue one. Hence this is in R iff $D^{-1}\vec{a}$ is parallel to the axis of rotation of D'. But this cannot possibly be true for all $\vec{a}\in\mathbb{R}^3$ and $D\in SO(3)$; hence R is not invariant.

Note that for fixed $(\vec{a}, D)$ the subset

$$g(a,D)\, R\, [g(a,D)]^{-1}$$
$$= \{\, g(\vec{a} - DD'D^{-1}\vec{a}, DD'D^{-1}) : D'\in SO(3)\}$$

is again a group:

$$g(\vec{a} - DD'_1D^{-1}\vec{a}, DD'_1D^{-1}) \times$$
$$g(\vec{a} - DD'_2D^{-1}\vec{a}, DD'_2D^{-1})$$

$$= g(\vec{a} - DD'_1D^{-1}\vec{a} + DD'_1D^{-1}(\vec{a} - DD'_2D^{-1}\vec{a}),$$
$$DD'_1D'_2D^{-1})$$

$$= g(\vec{a} - DD'_1D'_2D^{-1}\vec{a}, DD'_1D'_2D^{-1})$$

Conjugation of R by $g(\vec{0}, D)$ clearly just gives back R. But conjugation with $g(\vec{a}, E_3)$ gives

$$g(\vec{a}, E_3)\, R\, [g(\vec{a}, E_3)]^{-1}$$

$$= \{ g(\vec{a} - D'\vec{a}, D') : D' \in SO(3) \}$$

which is again a subgroup isomorphic to R. It represents rotations not about the origin (as R does) but about the point $\vec{a}$.

Problem 4

In this example, the CMB background plays the analogous rôle as the air in Galileo's example with the ship. Closing the shutters and keeping the air (wind) out of the rooms under deck corresponds then to isolating the laboratory from CMB radiation, i.e. by conducting walls. The relativity principle will continue to hold for all these local systems effectively decoupled from the CMB.