Sheet 10 : Solutions

Problem 1

In time intervall dt an amount dm_2 of rest mass is added to the system "gas" which moves with velocity β_2 relative to the fixed inertial system. The standard velocity-addition formula gives

$$\beta_2 = \frac{\beta_1 - \beta}{1 - \beta_1 \beta} \tag{10.1.1}$$

since β_1 = velocity of rocket relative to fixed inertial system

$-\beta$ = velocity of gas relative to rocket in same direction as β_2.

Hence the energy and momentum gain of system "gas" during dt are

$$\gamma_2 \, dm_2 \quad \text{and} \quad \gamma_2 \beta_2 \, dm_2 \tag{10.1.2}$$

Conservation of total energy and momentum then lead to

$$d(m_1 \gamma_1) = - \gamma_2 \, dm_2 \tag{10.1.3}$$

$$d(m_1 \gamma_1 \beta_1) = - \gamma_2 \beta_2 \, dm_2 \tag{10.1.4}$$

From (10.1.3) get

$$d m_2 = -\frac{1}{\gamma_2} \, d(m_1 \gamma_1) \qquad (10.1.5)$$

Insert this and β_2 of (10.1.1) into (10.1.3) and get

$$d(m_1 \gamma_1 \beta_1) = -\gamma_2 \, \frac{\beta_1 - \beta}{1 - \beta_1 \beta} \left(-\frac{1}{\gamma_2}\right) d(m_1 \gamma_1)$$

$$= \frac{\beta_1 - \beta}{1 - \beta_1 \beta} \, d(m_1 \gamma_1) \qquad (10.1.6)$$

$$\Leftrightarrow \quad m_1 \gamma_1 \, d\beta_1 = \left[\frac{\beta_1 - \beta}{1 - \beta_1 \beta} - \beta_1\right] d(m_1 \gamma_1)$$

$$= \frac{-\beta (1 - \beta_1^2)}{1 - \beta_1 \beta} \, d(m_1 \gamma_1)$$

$$= \frac{-\beta \, \gamma_1^{-2}}{1 - \beta_1 \beta} \, d(m_1 \gamma_1) \qquad (10.1.7)$$

We could, at this point, seperate the Variables into β_1 and $(m_1 \gamma_1)$. However, it turns out to be easier to seperate into β_1 and m_1. To do this we have to bring the $d\gamma_1$-term from the right-hand side of (10.1.7) to the left-hand side. Note that

$$d\gamma_1 = d(1 - \beta_1^2)^{-1/2} = \gamma^3 \, \beta_1 \, d\beta_1 \qquad (10.1.8)$$

Then (10.1.7) turns into

$$\underbrace{\left[m_1\gamma_1 + \frac{\beta\gamma_1^{-2}}{1-\beta_1\beta}\, m_1\gamma_1^3\beta_1\right]}\, d\beta_1 = -\frac{\beta\gamma_1^{-1}}{1-\beta_1\beta}\, dm_1$$

$$m_1\gamma_1\left[1+\frac{\beta\beta_1}{1-\beta_1\beta}\right] = \frac{m_1\gamma_1}{1-\beta_1\beta}$$

Hence (10.1.7) is equivalent to

$$m_1\gamma_1\, d\beta_1 = -\beta\gamma_1^{-1}\, dm_1$$

or
$$-\frac{1}{\beta}\,\frac{d\beta_1}{1-\beta_1^2} = \frac{dm_1}{m_1}$$

or
$$-\frac{1}{2\beta}\left(\frac{1}{1-\beta_1}+\frac{1}{1+\beta_1}\right)d\beta_1 = \frac{dm_1}{m_1} \qquad (10.1.9)$$

Integration between the initial velocity $(\beta_1)_{in.} = 0$ and initial mass $(m_1)_{in.} = M$, and final values β_1 and m_1, respectively, we get

$$-\frac{1}{2\beta}\ln\left(\frac{1+\beta_1}{1-\beta_1}\right) = \ln\left(\frac{m_1}{M}\right) \qquad (10.1.10)$$

or
$$\left(\frac{1-\beta_1}{1+\beta_1}\right)^{\frac{1}{2\beta}} = \frac{m_1}{M}$$

or
$$\frac{1-\beta_1}{1+\beta_1} = \left(\frac{m_1}{M}\right)^{2\beta}$$

or

$$\beta_1 = \frac{1-\left(\frac{m_1}{M}\right)^{2\beta}}{1+\left(\frac{m_1}{M}\right)^{2\beta}} \tag{10.1.11}$$

or

$$\beta_1 = \frac{\left(\frac{M}{m_1}\right)^{\beta}-\left(\frac{M}{m_1}\right)^{-\beta}}{\left(\frac{M}{m_1}\right)^{\beta}+\left(\frac{M}{m_1}\right)^{-\beta}} \tag{10.1.12}$$

Hence

$$\beta_1^2 = \frac{\left(\frac{M}{m_1}\right)^{2\beta}+\left(\frac{M}{m_1}\right)^{-2\beta}-2}{\left(\frac{M}{m_1}\right)^{2\beta}+\left(\frac{M}{m_1}\right)^{-2\beta}+2}$$

$$1-\beta_1^2 = \frac{4}{\left[\left(\frac{M}{m_1}\right)^{\beta}+\left(\frac{M}{m_1}\right)^{-\beta}\right]^2}$$

$$\Rightarrow \gamma_1 = \frac{1}{2}\left[\left(\frac{M}{m_1}\right)^{\beta}+\left(\frac{M}{m_1}\right)^{-\beta}\right]$$

$$= \frac{1}{2}\left[\left(\frac{m_1}{M}\right)^{\beta}+\left(\frac{m_1}{M}\right)^{-\beta}\right] \tag{10.1.13}$$

This relation can also be inverted, writing $X := (m_1/M)^\beta$ it leads

$$\gamma_1 = \frac{1}{2}\left(X + X^{-1}\right)$$

$$\Leftrightarrow \quad X^2 - 2X\gamma_1 + 1 = 0 \tag{10.1.14}$$

of which we seek the root $X < 1$

The two roots of (10.1.14) are

$$X_{1,2} = \gamma_1 \pm (\gamma_1^2 - 1)^{1/2}$$

so our root is

$$\left(\frac{m_1}{M}\right)^{\beta} = \gamma_1 - (\gamma_1^2 - 1)^{1/2}$$

or $$\frac{m_1}{M} = \left[\gamma_1 - (\gamma_1^2 - 1)^{1/2}\right]^{1/\beta} \qquad (10.1.15)$$

This equation tells you how what fraction of the initial rest mass M you have to burn in order to reach a γ-factor γ_1.

Equation (10.1.12) is called the "Relativistic Tsiolkovski-Equation". It's non-relativistic version is obtained for $c \to \infty$. Note that with $\beta_1 = v_1/c$ and $\beta = w/c$ it reads

$$v_1 = c \cdot \frac{1 - \left(\frac{m_1}{M}\right)^{2w/c}}{1 + \left(\frac{m_1}{M}\right)^{2w/c}} \qquad (10.1.16)$$

In order to take $c \to \infty$ limit we write

$$\left(\frac{m_1}{M}\right)^{2w/c} = \exp\left(\frac{2w}{c} \ln\left(\frac{m_1}{M}\right)\right) \qquad (10.1.17)$$

and expand

$$\exp\left(\frac{2W}{c}\ln\left(\frac{m_1}{M}\right)\right)$$

$$= 1 + \frac{2W}{c}\ln\frac{m_1}{M} + O(c^{-2}) \qquad (10.1.18)$$

Then (10.1.16) becomes

$$V_1 = W\ln\left(\frac{M}{m_1}\right) + O(c^{-1}). \qquad (10.1.19)$$

This is the original Tsiolkowski eqn.
– a surprisingly simple equation!

3.) The distance to the Andromeda galaxy is 2.5×10^6 ly. In order to get there in 80 years we need to reach a γ-factor of about

$$\gamma = 2.5\times 10^6 / 80 = 3.13 \times 10^4 \qquad (10.1.20)$$

According to (10.1.15) this gives

$$\frac{m_1}{M} = \left\{3.13\times10^4 - \left[(3.13\times10^4)^2 - 1\right]^{\frac{1}{2}}\right\}^{1/\beta}$$

$$= \left(1.6 \times 10^{-5}\right)^{1/\beta} \qquad (10.1.21)$$

Now, $\beta = \frac{W}{c}$, W = ejection velocity of gas

So $\frac{1}{\beta} \geq 1$. The bigger $\frac{1}{\beta}$ is, the smaller $(1.6 \times 10^{-5})^{1/\beta}$ becomes (as the number in brackets is < 1). So m_1 is largest for the smallest possible $\frac{1}{\beta}$ i.e. the largest possible w, which is $w = c$, $\beta = 1$. Hence, in order to reach Andromeda you have to burn <u>at least</u>

$$\frac{M - m_1}{M} \times 100 = (1 - 1.6 \times 10^{-5}) \times 100$$

$$= 99.9984\ \% \qquad (10.1.22)$$

of your spaceship.

The kinetic energy of a small cosmic grain of dust of mass $\mu g = 10^{-9}$ kg at $\gamma = 3.13 \times 10^4$ is

$$E_{dust} = 10^{-9}\,\mathrm{kg}\, c^2 \,(3.13 \times 10^4 - 1)$$

$$= 2.81 \times 10^{12}\ \mathrm{J} \qquad (10.1.23)$$

For that to equal the kinetic energy of a car of mass $m_c = 10^3$ kg, the γ-factor of the car must satisfy

$$m_{dust}\, c^2 (\gamma_{dust} - 1) = m_{car}\, c^2 (\gamma_{car} - 1)$$

$$\curvearrowright \quad \gamma_{car} = 1 + \frac{m_{dust}}{m_{car}} (\gamma_{dust} - 1)$$

$$= 1 + \frac{10^{-9}}{3 \cdot 10^3} (3.13 \times 10^4 - 1)$$

$$= 1 + 1.04 \times 10^{-8} \qquad (10.1.24)$$

This corresponds to a velocity of the car of

$$V_{car} = c\,(1 - \gamma_{car}^{-2})^{1/2}$$

$$= c\left[1 - (1 + 1.04 \times 10^{-8})^{-2}\right]^{1/2}$$

$$\cong c\left[2.08 \times 10^{-8}\right]^{1/2}$$

$$= c \times 1.44\ 10^{-4}$$

$$= 4.3\ 10^{4}\ \frac{m}{s}$$

$$= 1.5 \times 10^{5}\ km/h \qquad (10.1.22)$$

Even if we wanted to get to $\gamma = 2, \ldots$ so as to roughly half the eigentime in years than the distance in ly, we had

$$\gamma_{car} = 1 + \frac{m_{dust}}{m_{car}} = 1 + 3.3 \times 10^{-13} \qquad (10.1.23)$$

$$\rightsquigarrow V_{car} = c\left[1 - (1 + 3.3 \times 10^{-13})^{-2}\right]^{1/2}$$

$$= c\left[6.6 \times 10^{-13}\right]^{1/2}$$

$$= c \times 8.12 \times 10^{-7}$$

$$= 2.4 \times 10^{2}\ m/s$$

$$= 8.76 \times 10^{2}\ km/h \qquad (10.1.24)$$

almost sonic speed!

Problem 2

$$L(\vec{x},\dot{\vec{x}}) = -mc^2\left(1-\frac{\dot{\vec{x}}^2}{c^2}\right)^{1/2} + eEx \quad (10.2.1)$$

$$\frac{\partial L}{\partial \dot{\vec{x}}} = \gamma m \dot{\vec{x}} \quad , \quad \gamma = \left(1-\frac{\dot{\vec{x}}^2}{c^2}\right)^{-1/2} \quad (10.2.2)$$

$$\frac{\partial L}{\partial \vec{x}} = eE\vec{e}_x \quad (10.2.3)$$

$\Rightarrow$ Euler-Lagrange equations

$$\frac{d}{dt}(\gamma\dot{\vec{x}}) = \frac{e}{m}E\vec{e}_x \quad (10.2.4)$$

Integration $\int_0^t dt'$ with $\dot{\vec{x}}(0) = v_0\vec{e}_y$ gives

$$\gamma\dot{\vec{x}}(t) = \frac{e}{m}Et\,\vec{e}_x + \gamma_0 v_0\vec{e}_y \quad (10.2.5)$$

where $\gamma_0 := (1-v_0^2/c^2)^{-1/2}$

Note that equations 1-4 are just the same as (9.5.1-4); only 5. differs from (9.5.5) by the term $\gamma_0 v_0 \vec{e}_y$.

We set

$$\dot{\vec{x}} = \dot{x}\vec{e}_x + \dot{y}\vec{e}_y \quad (10.2.6)$$

and get the 2-component equivalent to (10.2.5):

$$\gamma \dot{x} = \frac{e}{m} E t \qquad (10.2.7)$$

$$\gamma \dot{y} = \gamma_0 v_0 \qquad (10.2.8)$$

where $\gamma = \left[1 - (\dot{x}^2 + \dot{y}^2)/c^2\right]^{-1/2}$

because γ contains $\dot{x}$ as well as $\dot{y}$ (7) and (8) are not yet decoupled (as it was the case for $v_0 = 0$).

In order to decouple them, we divide (7) by (8) and get:

$$\dot{x} = \frac{eE}{m\gamma_0 v_0} t \; \dot{y} . \qquad (10.2.9)$$

Next we square (8):

$$\dot{y}^2 = \gamma_0^2 v_0^2 \left[1 - (\dot{x}^2 + \dot{y}^2)/c^2\right] \qquad (10.2.10)$$

and replace $\dot{x}$ on the right-hand side via (9):

$$\dot{y}^2 = (\gamma_0 v_0)^2 \left[1 - \frac{\dot{y}^2}{c^2}\left(1 + \left(\frac{eE}{m\gamma_0 v_0}\right)^2 t^2\right)\right] \qquad (10.2.11)$$

which we can solve for $\dot{y}$

$$\dot{y}^2 = \frac{(\gamma_0 v_0)^2}{1 + (\gamma_0 \beta_0)^2 + \left(\frac{eE}{mc} t\right)^2} \qquad (10.2.12)$$

or

$$\dot{y} = \frac{v_0}{\sqrt{1 + \left(\frac{eE}{\gamma_0 mc} t\right)^2}} \qquad (10.2.13)$$

Where we used

$$1+(\gamma_0\beta_0)^2 = 1+\frac{\beta_0^2}{1-\beta_0^2} = \gamma_0^2 \qquad (10.2.14)$$

Re-inserting (13) into (9) gives

$$\dot{x} = \frac{eEt/m\gamma_0}{\sqrt{1+\left(\frac{eEt}{\gamma_0 mc}\right)^2}} \qquad (10.2.15)$$

Equations (13) and (15) are now decoupled and can be integrated. Setting $x(0)=y(0)=0$ we get

$$x(t) = \int_0^t dt' \frac{eEt'/m\gamma_0}{\sqrt{1+\left(\frac{eEt'}{\gamma_0 mc}\right)^2}}$$

$$= c\,\frac{\gamma_0 mc}{eE}\left\{\sqrt{1+\left(\frac{eEt}{\gamma_0 mc}\right)^2}-1\right\}$$

$$= c\tau\left\{\sqrt{1+\left(\frac{t}{\tau}\right)^2}-1\right\} \qquad (10.2.16)$$

with $\tau := \dfrac{\gamma_0 mc}{eE}$ (10.2.17)

And further

$$y(t) = \int_0^t dt' \frac{v_0}{\sqrt{1+\left(\frac{t}{\tau}\right)^2}} = v_0\tau \,\mathrm{arsinh}\left(\frac{t}{\tau}\right) \qquad (10.2.18)$$

S 10.12

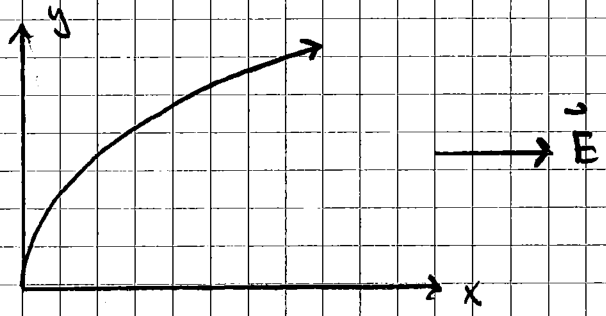
We can solve (18) for $t = t(y)$:

$$t = \tau \sinh(y / v_0 \tau) \qquad (10.2.19)$$

Inserting this into (16) gives x as function of y

$$x(y) = c\,\tau\left[\cosh(y/v_0\tau) - 1\right] \qquad (10.2.20)$$

which is a catenary ("Kettenlinie")



(10.2.21)

In the non-relativistic approximation $c \to \infty$ have $\tau := \frac{\gamma_0 m c}{eE} \to \infty$ and $(1/\tau) \to 0$. Expanding (20) in the Argument $(y/v_0\tau)$ around zero gives in leading order

$$\cosh(y/v_0\tau) - 1 = \frac{1}{2}(y/v_0\tau)^2 + \dots$$

$$x(y) = \frac{1}{2}\frac{y^2}{v_0^2\tau^2}c\,\tau + \dots$$

$$= \frac{1}{2}\frac{c}{\tau}(y/v_0)^2 = \frac{1}{2}\frac{eE}{m v_0^2}y^2 \qquad (10.2.22)$$

where we also used $\gamma_0 = 1$ for $c \to \infty$.

This is a parabola, as expected!

Note that from (16) and (18) we also easily get the non-relativistic limits of $X(t)$ and $y(t)$:

$$X(t) = c\tau \, \frac{1}{2}\left(\frac{t}{\tau}\right)^2 + \cdots$$

$$= \frac{1}{2} t^2 \frac{c}{\tau} + \cdots$$

$$= \frac{1}{2} \frac{eE}{m} t^2 \qquad (10.2.23)$$

$$y(t) = v_0 \tau \left(\frac{t}{\tau}\right) + \cdots$$

$$= v_0 t \qquad (10.2.24)$$

This is the classical "constant-force-field-motion".

3) The inertial time t it takes to $X = h$ follows from (16)

$$h = c\tau\left[\left(1+\left(\frac{t}{\tau}\right)^2\right)^{1/2} - 1\right] \quad \Rightarrow$$

$$\left\{\left[\left(\frac{h}{c\tau}\right) + 1\right]^2 - 1\right\}^{1/2} \tau = t(h) \qquad (10.2.25)$$

In the non-relativistic limit $\tau = \frac{\gamma_0 mc}{eE} \to \infty$ and we get

$$t(h) = \left(\frac{2hm}{eE}\right)^{1/2} \qquad (10.2.26)$$

which is independent of v_0. This means that in the non-relativistic case the time it needs for the particle to move into the direction of the force field by a fixed distance h does not depend on the initial velocity perpendicular to that direction. However, in the relativistic case this is not true anymore since τ depends on γ_0

$$t(h) = \frac{\gamma_0 mc}{eE}\left\{\left[\frac{heE}{\gamma_0 mc^2}+1\right]^2-1\right\}^{1/2} \qquad (10.2.27)$$

For $\gamma_0 \to \infty$ the leading order of this is

$$t(h) = \frac{\gamma_0 mc}{eE}\left(\frac{2heE}{\gamma_0 mc^2}\right)^{1/2}$$

$$= \gamma_0^{1/2}\left(\frac{2hm}{eE}\right)^{1/2} \qquad (10.2.28)$$

which is $\gamma_0^{1/2}$-times the classical "free-fall time" (26) and diverges to ∞ for $\gamma_0 \to \infty$. So relativistic effects enhance the "free-fall time" t, which is sometimes expressed by saying that the initial transversal velocity v_0 enhances the inertial mass m to $\gamma_0 m$.

What about proper time? The relation between proper time σ (we already used τ, so proper time is now called σ) and inertial time t is

$$d\sigma = (1-\beta^2)^{1/2}\, dt \qquad (10.2.29)$$

and

$$\beta^2 = (\dot{x}^2 + \dot{y}^2)/c^2 \qquad (10.2.30)$$

From (13) and (15) have

$$\dot{x} = \frac{ct/\tau}{\sqrt{1+(t/\tau)^2}} \qquad (10.2.31)$$

$$\dot{y} = \frac{v_0}{\sqrt{1+(t/\tau)^2}} \qquad (10.2.32)$$

$$\dot{x}^2 + \dot{y}^2 = \frac{(ct/\tau)^2 + v_0^2}{1+(t/\tau)^2} \Rightarrow$$

$$1 - \frac{\dot{x}^2+\dot{y}^2}{c^2} = \frac{\gamma_0^{-2}}{1+(t/\tau)^2} \qquad (10.2.33)$$

Hence

$$d\sigma = \gamma_0^{-1} \frac{dt}{\sqrt{1+(t/\tau)^2}}$$

$$= \frac{\tau}{\gamma_0} \frac{d(t/\tau)}{\sqrt{1+(t/\tau)^2}} \qquad (10.2.34)$$

and integration leads to

$$\sigma(t) = \frac{\tau}{\gamma_0} \operatorname{arsinh}\left(\frac{t}{\tau}\right) \tag{10.2.35}$$

or

$$t = \tau \sinh\left(\gamma_0 \frac{\sigma}{\tau}\right) \tag{10.2.36}$$

Hence we can express x and y in terms of σ (rather than t):

$$x(\sigma) = c\tau \left\{\cosh\left(\gamma_0 \frac{\sigma}{\tau}\right) - 1\right\}$$

$$= \frac{\gamma_0 mc^2}{eE} \left\{\cosh\left(\frac{eE}{mc}\sigma\right) - 1\right\} \tag{10.2.37}$$

$$y(\sigma) = v_0 \tau \frac{\sigma \gamma_0}{\tau} = v_0 \gamma_0 \tau \tag{10.2.38}$$

The proper time from $x = 0$ to $x = h$ is

$$\sigma = \frac{\tau}{\gamma_0} \operatorname{arcosh}\left(1 + \frac{h}{c\tau}\right)$$

$$= \frac{mc}{eE} \operatorname{arcosh}\left(1 + \frac{heE}{\gamma_0 mc^2}\right) \tag{10.2.39}$$

For small x have

$$\operatorname{arcosh}(1+x) = \sqrt{2}\sqrt{x} - \frac{x^{3/2}}{6\sqrt{2}} + \dots \qquad 10.2.40$$

hence to leading order in $\frac{heE}{\gamma_0 mc^2}$

$$\sigma = \frac{mc}{eE}\left(2\frac{heE}{\gamma_0 mc^2}\right)^{1/2} + \dots$$

$$= \gamma_0^{-1/2}\left(\frac{2hm}{eE}\right)^{1/2} \qquad (10.2.41)$$

Notice that here γ_0 enters inversely as compared to (10.2.28). The coordinate time $t(h)$ grows with γ_0 like $\gamma_0^{1/2}$, the proper time $\tau(h)$ drops with γ_0 like $\gamma_0^{-1/2}$.

Problem 3

1.) If $u_{1,2}$ are two states of motion (timelike, future-oriented, normalised vectors in V), then their γ-factor is

$$\gamma = (1-\beta^2)^{-1/2} = u_1 \cdot u_2 \qquad (10.3.1)$$

where $\beta = \frac{v}{c}$ is the modulus of the relative velocity of β_1 relative to β_2 or vice versa. Hence

$$1-\beta^2 = \left(\frac{1}{u_1 \cdot u_2}\right)^2, \quad \text{or}$$

$$\beta = \frac{\sqrt{(u_1 \cdot u_2)^2 - 1}}{u_1 \cdot u_2} \qquad (10.3.2)$$

2.) If

$$u_1 = \gamma_1 (e_0 + \beta_1^k e_k) \qquad (10.3.3a)$$

$$u_2 = \gamma_2 (e_0 + \beta_2^k e_k) \qquad (10.3.3b)$$

where $e_\alpha \cdot e_\beta = \eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$ then

$$u_1 u_2 = \gamma_1 \gamma_2 (1 - \vec{\beta}_1 \cdot \vec{\beta}_2) \qquad (10.3.4)$$

and

$$\beta = \frac{\left[(1-\vec{\beta}_1 \cdot \vec{\beta}_2)^2 - \gamma_1^{-2} \gamma_2^{-2}\right]^{1/2}}{1-\vec{\beta}_1 \cdot \vec{\beta}_2} \qquad (10.3.5)$$

Now,

$$(1-\beta_1\cdot\beta_2)^2 - (1-\beta_1^2)(1-\beta_2^2)$$

$$= 1 - 2\,\beta_1\cdot\beta_2 + (\vec{\beta}_1\cdot\vec{\beta}_2)^2$$

$$\quad - 1 + \beta_1^2 + \beta_2^2 - \beta_1^2\,\beta_2^2$$

$$= (\beta_1 - \beta_2)^2 - \left(\beta_1^2\,\beta_2^2 - (\vec{\beta}_1\cdot\vec{\beta}_2)^2\right)$$

$$= (\beta_1 - \beta_2)^2 - (\vec{\beta}_1 \times \vec{\beta}_2)^2 . \qquad (10.3.6)$$

Hence

$$\beta = \frac{\left[(\vec{\beta}_1 - \vec{\beta}_2)^2 - (\vec{\beta}_1 \times \vec{\beta}_2)^2\right]^{1/2}}{1 - \vec{\beta}_1\cdot\vec{\beta}_2} \qquad (10.3.7)$$

Problem 4

$$\gamma : \mathbb{R} \to M \tag{10.4.1}$$

Parametrization with respect to proper time means that

$$\dot\gamma \cdot \dot\gamma = c^2 \tag{10.4.2}$$

1) $$\rightsquigarrow \quad \dot\gamma \cdot \ddot\gamma = 0 \tag{10.4.3}$$

2) The projector $P^\perp_u$ orthogonal to a timelike unit vector u is

$$P^\perp_u = id|_V - u \otimes u_\downarrow \tag{10.4.4}$$

where $u_\downarrow = \eta(u, \cdot)$.

With $u = \dot\gamma / c$ get

$$P^\perp_{\dot\gamma}(\dddot\gamma) = \dddot\gamma - \dot\gamma \, \frac{\dot\gamma \cdot \dddot\gamma}{c^2} = 0 \tag{10.4.5}$$

It means that the variation of $\ddot\gamma$ = proper acceleration in the instantaneous rest-frame (i.e. the orthogonal complement of $\dot\gamma$) vanishes.

Scalar multiplication of (10.4.5) with $\ddot\gamma$ gives, in view of (10.4.3):

$$\dddot{\gamma} \cdot \ddot{\gamma} = 0 \qquad (10.4.6)$$

or
$$\ddot{\gamma}^2 = \text{const.} \qquad (10.4.7)$$

Now,
$$\dot{\gamma} \cdot \dddot{\gamma} = (\dot{\gamma} \cdot \ddot{\gamma})^{\cdot} - \ddot{\gamma}^2 = -\ddot{\gamma}^2 \qquad (10.4.8)$$

Hence (10.4.5) is equivalent to
$$\dddot{\gamma} = \dot{\gamma}\, \omega^2 \qquad (10.4.9)$$

where
$$\omega := \left(-\ddot{\gamma}^2 / c^2\right)^{1/2} \qquad (10.4.10)$$

Note that $\dot{\gamma} \cdot \ddot{\gamma} = 0$ implies that $\ddot{\gamma}$ is spacelike so that $\ddot{\gamma}^2 \leq 0$

3) Constancy of ω implies that the most general integral of (10.4.9) is

$$\dot{\gamma}(\tau) = a \exp(\omega \tau) + b \exp(-\omega\tau) \qquad (10.4.11)$$

$$\curvearrowright \gamma(\tau) = p + l_1 \exp(\omega\tau) + l_2 \exp(-\omega\tau) \qquad (10.4.12)$$

with $l_1 = a/\omega$, $l_2 = b/\omega$

and $l_1, l_2, a, b \in V$.

But we also have to satisfy (10.4.10):

$$\ddot{\gamma} = \omega^2 l_1 \exp(\omega\tau) - \omega^2 l_2 \exp(-\omega\tau) \qquad (10.4.13)$$

$$\rightsquigarrow \ddot{\gamma}^2 = \omega^4 (l_1 \cdot l_1) \exp(2\omega\tau) + \omega^4 (l_2 \cdot l_2) \exp(-2\omega\tau) + 2\omega^4 l_1 \cdot l_2 \qquad (10.4.14)$$

which, according to (10.4.10), must equal $-c^2\omega^2$. This is the case if and only if

$$l_1 \cdot l_1 = l_2 \cdot l_2 = 0 \qquad (10.4.15)$$

$$\text{and} \quad l_1 \cdot l_2 = \frac{c^2}{2\omega^2} \qquad (10.4.16)$$

(10.4.15) means that l_1 and l_2 are lightlike. Hence they span a timelike plane. In that plane we can introduce orthonormal vectors

4)
$$e_0 := \frac{\omega}{c}(l_1 + l_2) \qquad (10.4.17a)$$

$$e_1 := \frac{\omega}{c}(l_1 - l_2) \qquad (10.4.17b)$$

then
$$e_0^2 = \frac{\omega^2}{c^2}\, 2\, l_1 \cdot l_2 = 1 \qquad (10.4.18a)$$

$$e_1^2 = \frac{\omega^2}{c^2}(-2\, l_1 \cdot l_2) = -1 \qquad (10.4.18b)$$

$$e_0 \cdot e_1 = \frac{\omega^2}{c^2}(l_1^2 - l_2^2) = 0. \qquad (10.4.18c)$$

Hence (10.4.12) reads

$$\gamma(\tau) = p + \frac{c}{\omega}\left(\sinh(\omega\tau)\, e_0 + \cosh(\omega\tau)\, e_1\right) \qquad (10.4.19)$$

Note that

$$a := \left(-\ddot{\gamma}^2\right)^{1/2} = c\,\omega \qquad (10.4.20)$$

is the modulus of the four-acceleration, hence

$$\gamma(\tau) = p + \frac{c^2}{a}\left[\sinh\left(\frac{a}{c}\tau\right)e_0 + \cosh\left(\frac{a}{c}\tau\right)e_1\right] \qquad (10.4.21)$$

or

$$\gamma(s) = p + \frac{c^2}{a}\left[\sinh\left(\frac{a}{c^2}s\right)e_0 + \cosh\left(\frac{a}{c^2}s\right)e_1\right] \qquad (10.4.22)$$

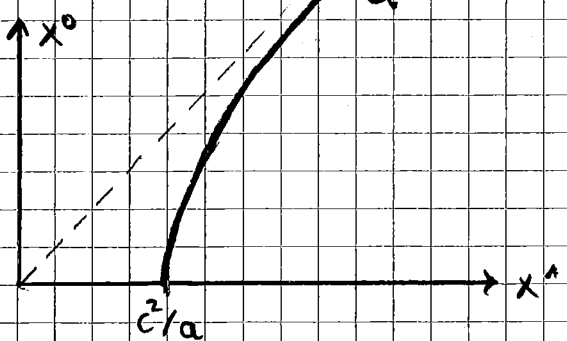
where $s = c\tau$ is proper length

This is a hyperbola in the plane spanned by e_0 and e_1. If we set $p = 0$ (origin) and introduce coordinates x^0 and x^1 then

$$x^0(s) = \frac{c^2}{a}\sinh\left(\frac{a}{c^2}s\right) \qquad (10.4.23)$$

$$x^1(s) = \frac{c^2}{a}\cosh\left(\frac{a}{c^2}s\right) \qquad (10.4.24)$$

$$\rightsquigarrow\ (x^1)^2 - (x^0)^2 = \frac{c^2}{a} \qquad (10.4.25)$$



Problem 5

1)
$$d\tau = \frac{1}{c}\,ds = \frac{1}{c}\left[(\dot{x}^0)^2 - \dot{\vec{x}}^2\right]^{1/2} d\lambda \qquad (10.5.1)$$

Choose $\lambda = t \rightsquigarrow \dot{x}^0 = c$, then, using

$$\frac{\dot{\vec{x}}}{\dot{x}^0} = \frac{1}{c}\frac{d\vec{x}}{dt} \qquad (10.5.2)$$

$$d\tau = (1-\vec{\beta}^2)^{1/2}\,dt = \gamma^{-1}\,dt$$

or
$$\frac{d}{d\tau} = \gamma\,\frac{d}{dt} \qquad (10.5.3)$$

2) Write $\dot{} = \frac{d}{d\tau}$, $' = \frac{d}{dt}$

$$\rightsquigarrow \quad \dot{x}^\alpha = \gamma\,\frac{d}{dt}(ct, \vec{x})$$
$$= \gamma(c, \vec{x}') = \gamma(c, \vec{v}) \qquad (10.5.4)$$

Also
$$\gamma' = \frac{d}{dt}(1-\vec{\beta}^2)^{-1/2} = (1-\vec{\beta}^2)^{-3/2}\,\frac{\vec{v}\cdot\vec{v}'}{c^2}$$
$$= \gamma^3\,\frac{\vec{v}\cdot\vec{a}}{c^2} \qquad (10.5.5)$$

$$\ddot{x}^\alpha = \gamma(\dot{x}^\alpha)' = \gamma\,[\gamma(c,\vec{v})]'$$
$$= \gamma'\,\dot{x}^\alpha + \gamma^2(0,\vec{a}) \qquad (10.5.6)$$

$$\dddot{x}^\alpha = \gamma\gamma'' \dot{x}^\alpha + \gamma' \ddot{x}^\alpha + 2\gamma^2\gamma'(0,\vec{a}) + \gamma^3(0,\vec{b})$$

$$= \gamma\gamma''\dot{x}^\alpha + \gamma'(\gamma'\dot{x}^\alpha + \gamma^2(0,\vec{a})) + 2\gamma^2\gamma'(0,\vec{a}) + \gamma^3(0,\vec{b})$$

$$= (\gamma\gamma')'\dot{x}^\alpha + 3\gamma'\gamma^2(0,\vec{a}) + \gamma^3(0,\vec{b}) \qquad (10.5.7)$$

Projecting perpendicular to $\dot{\gamma}$, i.e. $\dot{x}^\alpha$ eliminates the first term and gives for the sum of 2nd and 3rd

$$P^\perp_{\dot{\gamma}}\ (0,\ 3\gamma'\gamma^2\vec{a} + \gamma^3 b) \qquad (10.5.8)$$

In our coordinate representation

$$\dot{\gamma} = \dot{x}^\alpha = \gamma(c,\vec{v}) = c\ \underbrace{\gamma(1,\vec{\beta})}_{=:u} \qquad (10.5.9)$$

hence, if we abbreviate

$$3\gamma'\gamma^2\vec{a} + \gamma^3\vec{b} =: \vec{z}, \qquad (10.5.10)$$

then

$$P^\perp_u\ (0,\vec{z}) = (0,z) - u\,(u\cdot(0,z))$$

$$= (0,\vec{z}) - \gamma(1,\vec{\beta})\ \gamma[(1,\vec{\beta})\cdot(0,z)]$$

$$= (0,\vec{z}) - \gamma^2(1,\vec{\beta})\,(-\vec{\beta}\cdot\vec{z})$$

$$= \left(\gamma^2\, \vec{\beta}\cdot\vec{z}\ ,\ \vec{z} + \gamma^2 \vec{\beta}\,(\vec{\beta}\cdot\vec{z})\right) \qquad (10.5.11)$$

If that is to be the zero vector we must have

$$\vec{\beta}\cdot\vec{z} = 0 \qquad (10.5.12a)$$

and

$$\vec{z} + \gamma^2 \vec{\beta}\,(\vec{\beta}\cdot\vec{z}) = 0 \qquad (10.5.12b)$$

which is the case iff $\vec{z} = 0$.

Hence

$$P^{\perp}_{\dot\gamma}\,(0, 3\gamma'\gamma^2\vec{a} + \gamma^3\vec{b}) = 0$$

$$\Leftrightarrow\ 3\gamma'\gamma^2\vec{a} + \gamma^3\vec{b} = 0$$

$$\Leftrightarrow\ (\gamma^3\vec{a})' = 0 \qquad (10.5.13)$$

4.) Using (10.5.5) we have

$$(\gamma\vec{v})' = \gamma\vec{a} + \vec{v}\,\gamma^3(\vec{v}\cdot\vec{a})$$

$$= \gamma(\vec{a}_\perp + \vec{a}_{\parallel}) + \vec{v}\,\gamma^3(\vec{v}\cdot\vec{a}_{\parallel})/c^2$$

$$= \gamma\vec{a}_\perp + \gamma^3\left((1-\vec{\beta}^2)\vec{a}_{\parallel} + \vec{\beta}(\vec{\beta}\cdot\vec{a}_{\parallel})\right)$$

$$= \gamma\, a_\perp + \gamma^3\vec{a}_{\parallel} \qquad (10.5.14)$$

If (10.5.13) is integrated once to

$$\gamma^3 \vec{a} = \vec{g} = \overrightarrow{\text{const}} \tag{10.5.15}$$

and initial velocity $\vec{v}_0 \parallel \vec{g}$, then $\vec{a} = \vec{a}_\parallel$ and

$$\gamma^3 \vec{a} = \gamma^3 \vec{a}_\parallel = (\gamma \vec{v})' = \vec{g} \tag{10.5.16}$$

and the Problem is reduced to that of Problem 5 on Sheet 9.

If the initial velocity $\vec{v}_0$ is not parallel to $\vec{g}$ the motion will not be in a plane in spacetime, as we have seen in solution to Problem 2 above. Hence that motion is not of constant acceleration.