Sheet 11: Solutions

Problem 1

We start with a pure electric Coulomb-field

$$\vec{E}(t,\vec{x}) = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{x}}{r^3}, \quad r := \|\vec{x}\| \tag{11.1.1a}$$

$$B(t,\vec{x}) = \vec{0} \tag{11.1.1b}$$

Note that the dependence of $\vec{E}$ on t is trivial

A boost with velocity $\vec{v} = c\vec{\beta}$ has the effect

$$x^0 \to (Lx)^0 = \gamma\,(x^0 + \vec{\beta}\cdot\vec{x}) \tag{11.1.2a}$$

$$\vec{x} \to \overrightarrow{(L\vec{x})} = \vec{x}_\perp + \gamma\,(\vec{x}_\parallel + \vec{\beta}x^0) \tag{11.1.2b}$$

with inverse

$$x^0 \to (L^{-1}x)^0 = \gamma\,(x^0 - \vec{\beta}\cdot\vec{x}) \tag{11.1.3a}$$

$$\vec{x} \to \overrightarrow{(L^{-1}x)} = \vec{x}_\perp + \gamma\,(\vec{x}_\parallel - \vec{\beta}x^0) \tag{11.1.3b}$$

so that

$$\frac{\overrightarrow{(L^{-1}x)}}{\|\overrightarrow{(L^{-1}x)}\|^3} = \frac{\gamma\,(\vec{x}_\parallel - \vec{v}t) + \vec{x}_\perp}{\left[\gamma^2(\vec{x}_\parallel - \vec{v}t)^2 + \vec{x}_\perp^2\right]^{3/2}} \tag{11.1.4}$$

The transformation-laws for $\vec{E}$ and $\vec{B}$ are [Lecture 10, formulae (10.42)]

$$\vec{E}'_{\parallel}(x) = \vec{E}_{\parallel}(L^{-1}x) \qquad (11.1.5a)$$

$$\vec{E}'_{\perp}(x) = \gamma \left[\vec{E}_{\perp}(L^{-1}x) - \vec{v} \times \vec{B}(L^{-1}x) \right] \qquad (11.1.5b)$$

$$\vec{B}'_{\parallel}(x) = \vec{B}_{\parallel}(L^{-1}x) \qquad (11.1.6a)$$

$$\vec{B}'_{\perp}(x) = \gamma \left[B_{\perp}(L^{-1}x) + \frac{\vec{v}}{c^2} \times \vec{E}(L^{-1}x) \right] \qquad (11.1.6b)$$

For $\vec{B} \equiv \vec{0}$ this reduces to

$$\vec{E}'_{\parallel}(x) = \vec{E}_{\parallel}(L^{-1}x) \qquad (11.1.7a)$$

$$\vec{E}'_{\perp}(x) = \gamma \, \vec{E}_{\perp}(L^{-1}x) \qquad (11.1.7b)$$

$$\vec{B}'_{\parallel}(x) = \vec{0} \qquad (11.1.8a)$$

$$B'_{\perp}(x) = \gamma \frac{\vec{v}}{c^2} \times \underset{\uparrow}{\vec{E}}(L^{-1}x) = \frac{\vec{v}}{c^2} \times \vec{E}'_{\perp}(x) \qquad (11.1.8b)$$

here only $\vec{E}_{\perp}$ enters.

Now, using (11.1.1a), this gives

$$\vec{E}'_{\parallel}(t, \vec{x}) = \frac{Q}{4\pi\varepsilon_0} \frac{\overset{\downarrow}{\gamma}(\vec{x}_{\parallel} - \vec{v}t)}{\left[\gamma^2 (\vec{x}_{\parallel} - \vec{v}t)^2 + \vec{x}_{\perp}^2\right]^{3/2}} \qquad (11.1.9a)$$

$$\vec{E}'_{\perp}(t, \vec{x}) = \frac{Q}{4\pi\varepsilon_0} \overset{\downarrow}{\gamma} \frac{\vec{x}_{\perp}}{\left[\gamma^2 (\vec{x}_{\parallel} - \vec{v}t)^2 + \vec{x}_{\perp}^2\right]^{3/2}} \qquad (11.1.9b)$$

Note that the two γ's on the right-hand side marked by $\overset{\downarrow}{\gamma}$ have different origins:

The $\overset{\downarrow}{\gamma}$ on the right-hand side of (1.1.9a) originates from

$$(L^{-1}\vec{x})_{\parallel} = \overset{\downarrow}{\gamma}\,(\vec{x}_{\parallel} - \vec{v}\,t) \qquad (11.1.10)$$

whereas the $\overset{\downarrow}{\gamma}$ on the right-hand side of (1.1.9b) originates from that occurring in the transformation formula (1.1.7b). The <u>remarkable combined effect</u> of these two γ's is that (1.1.9a) and (1.1.9b) combine to

$$\vec{E}'(t,\vec{x}) = \frac{Q}{4\pi\varepsilon_0}\,\gamma\,\frac{(\vec{x} - \vec{v}t)}{\left[\gamma^2(\vec{x}_{\parallel} - \vec{v}t) + \vec{x}_{\perp}^2\right]^{3/2}} \qquad (11.1.11)$$

This is remarkable because it says that the electric field is radially pointing from the <u>instantaneous position</u> of the particle, which is

$$\vec{x}_0 = \vec{v}\,t \quad \text{(instantaneous position)} \qquad (11.1.12)$$

and <u>not</u> from the <u>retarded position</u>. And this despite the fact that the electromagnetic field can be derived from the <u>retarded potential</u> of a uniformly moving charge. This means that if the charge had changed its state of motion <u>after</u> it passed the retarded position of $\vec{x}$

the field would still be radially pointing from the instantaneous position that the charge would assume had it continued with constant velocity.

From the right hand side of (11.1.8b) we get ($c^2 \varepsilon_0 = 1/\mu_0$)

$$\vec{B}'(t,\vec{x}) = \frac{\mu_0}{4\pi} Q \gamma \frac{\vec{v}\times\vec{x}}{[\gamma^2(\vec{x}_\parallel - \vec{v}t)^2 + \vec{x}_\perp^2]^{3/2}} \qquad (11.1.13)$$

Note that because of

$$\gamma^2(\vec{x}_\parallel - \vec{v}t) + \vec{x}_\perp^2 = \gamma^2(\vec{x}-\vec{v}t)_\parallel^2 + (\vec{x}-\vec{v}t)_\perp^2 \qquad (11.1.14)$$

the dependence of $\vec{E}'$ and $\vec{B}'$ on t is just in the combination $(\vec{x}-\vec{v}t)$, which says that the field configuration at time t is identical to that at time 0, rigidly translated by $\vec{v}t$.
It is therefore sufficient to discuss the field at $t=0$:

$$\vec{E}'(\vec{x}) = \frac{Q}{4\pi\varepsilon_0} \gamma \frac{\vec{x}}{[\gamma^2 \vec{x}_\parallel^2 + \vec{x}_\perp^2]^{3/2}} , \qquad (11.1.15)$$

$$\vec{B}'(\vec{x}) = \frac{\mu_0 Q}{4\pi} \gamma \frac{\vec{v}\times\vec{x}}{[\gamma^2 \vec{x}_\parallel^2 + \vec{x}_\perp^2]^{3/2}} . \qquad (11.1.16)$$

$v \times \vec{x}$ is a field whose lines go in circles around $\vec{v}$-axis.

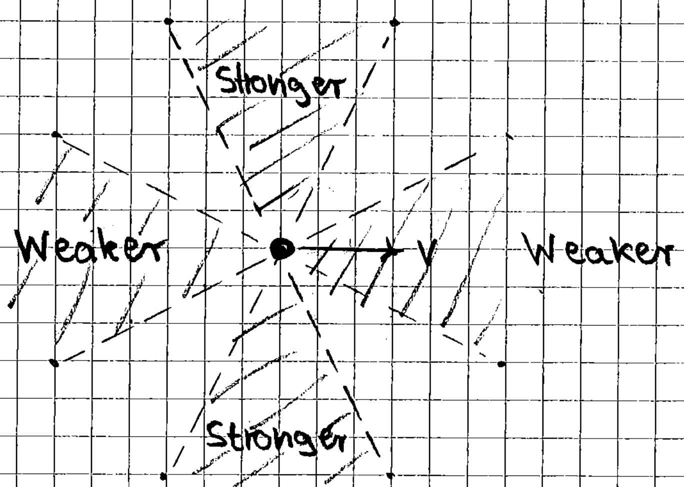
That means that for $\vec{x}_\perp = 0$, that is on the line of travel we have

$$\vec{E}(\vec{x})\Big|_{\vec{x}_\perp = 0} = \frac{Q}{4\pi\varepsilon_0}\,\gamma^{-2}\,\frac{\vec{x}}{r^3}\Big|_{\vec{x}_\perp = 0}\,, \tag{11.1.17}$$

a $\gamma^{-2} = (1-\beta^2)$ diminishment of the field strength as compared to the particle at rest, whereas for $\vec{x}_\parallel = \vec{0}$, i.e. on the plane perpendicular to the direction of travel and through the particle,

$$\vec{E}(\vec{x})\Big|_{\vec{x}_\parallel = 0} = \frac{Q}{4\pi\varepsilon_0}\,\gamma\,\frac{\vec{x}}{r^3}\Big|_{\vec{x}_\parallel = 0} \tag{11.1.18}$$

which corresponds to a $\gamma = (1-\beta^2)^{-1/2}$ enhancement.



(11.1.19)

An alternative representation is obtained by introducing spherical polar coordinates (r, θ, φ) with pole-axis in the direction of $\vec{v} = v\vec{e}_x$

$$\vec{X} = r \begin{pmatrix} \cos\theta \\ \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \end{pmatrix} = r\,\vec{n} \qquad (11.1.20)$$

$$\vec{X}_{\parallel} = r \begin{pmatrix} \cos\theta \\ 0 \\ 0 \end{pmatrix} \qquad (11.1.21)$$

$$\vec{X}_{\perp} = r \begin{pmatrix} 0 \\ \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \end{pmatrix} \qquad (11.1.21)$$

$$\vec{X}_{\parallel}^2 = r^2\cos^2\theta \;, \quad \vec{X}_{\perp}^2 = r^2\sin^2\theta$$

$$\gamma^2\vec{X}_{\parallel}^2 + X_{\perp}^2 = r^2\left[\gamma^2\cos^2\theta + \sin^2\theta\right]$$

$$= r^2\left[1 + (\gamma^2-1)\cos^2\theta\right] \qquad (11.1.22)$$

Hence

$$\vec{E}'(r,\theta) = \underbrace{\frac{\gamma}{\left[(\gamma^2-1)\cos^2\theta+1\right]^{3/2}}}_{\theta\text{-dependent scaling}} \; \underbrace{\frac{Q}{4\pi\varepsilon_0}\frac{\vec{n}}{r^2}}_{\text{ordinary Coulomb field}} \qquad (11.1.23)$$

$$= \underbrace{\frac{\gamma}{\left\{r^{4/3}\left[(\gamma^2-1)\cos^2\theta+1\right]\right\}^{3/2}}}_{(r,\theta)\text{-dep. magnitude}} \; \underbrace{\frac{Q}{4\pi\varepsilon_0}\vec{n}}_{\text{radial of const. magnitude}} \qquad (11.1.24)$$

And using

$$\vec{v} \times \vec{n} = \underbrace{\|\vec{v} \times \vec{n}\|}_{v \sin\theta} \; \underbrace{\frac{\vec{v} \times \vec{n}}{\|\vec{v} \times \vec{n}\|}}_{\vec{e}_\varphi}, \qquad (11.1.25)$$

$$\vec{B}'(r, \theta) = \frac{v \gamma \sin\theta}{\left\{ r^{4/3} \left[(\gamma^2 - 1) \cos^2\theta + 1 \right] \right\}^{3/2}} \frac{\mu_0 Q}{4\pi} \vec{e}_\varphi \qquad (11.1.26)$$

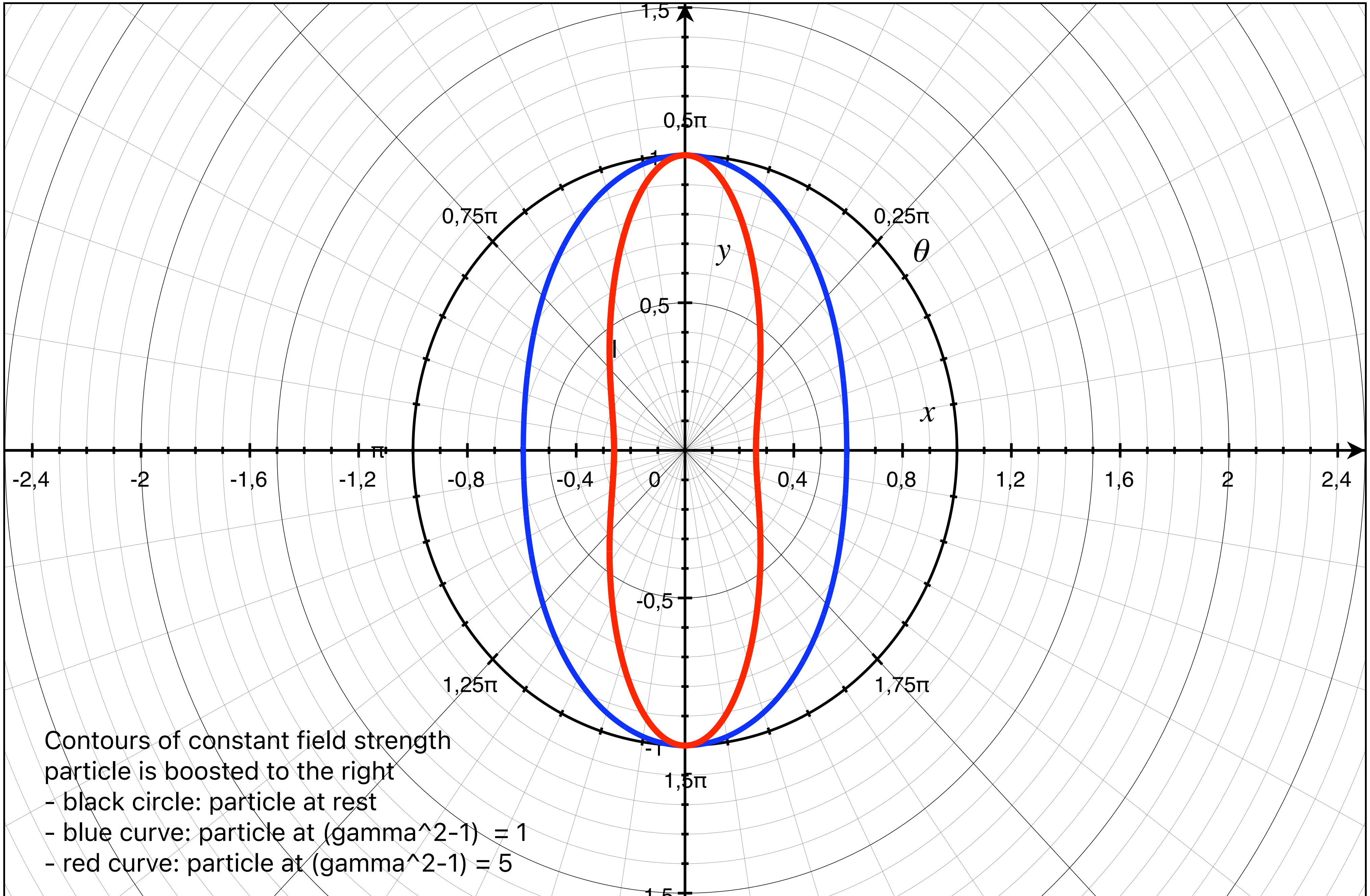
The electric field strength is constant if

$$r^{4/3} \left[(\gamma^2 - 1) \cos^2\theta + 1 \right] = k = \text{const} \qquad (11.1.27)$$

$$\Rightarrow \; r(\theta) = K / \left[1 + (\gamma^2 - 1) \cos^2\theta \right]^{3/4} \qquad (11.1.28)$$

(Where $K := k^{3/4} = \text{const}$.)

A plot of contours $r(\theta)$ for $K = 1$ and $\gamma = 1, \sqrt{2}, \sqrt{6}$ is given on the next page.


Contours of constant field strength
particle is boosted to the right
- black circle: particle at rest
- blue curve: particle at (gamma^2-1) = 1
- red curve: particle at (gamma^2-1) = 5
0,5π
0,25π
0,75π
1,25π
1,75π
1,5π
π
θ
x
y
-2,4
-2
-1,6
-1,2
-0,8
-0,4
0
0,4
0,8
1,2
1,6
2
2,4
1,5
1
0,5
-0,5
-1
-1,5

Part 2 of Problem 1

Since $A^\alpha = (\phi/c, \vec{A})$ transforms as a four vector and $\vec{A} = 0$ in rest-frame of charge, have

$$A'^\alpha(x) = L^\alpha{}_\beta\, A^\beta(y) = L^\alpha{}_0\, A^0(y) \qquad (11.1.29)$$

$$\Rightarrow \phi'(t,\vec{x}) = \frac{Q}{4\pi\varepsilon_0}\,\gamma\,\frac{1}{\left[\gamma^2(\vec{x}_\parallel - \vec{v}t)^2 + \vec{x}_\perp^2\right]^{1/2}} \qquad (11.1.30)$$

$$\vec{A}'(t,\vec{x}) = \frac{Q\mu_0}{4\pi}\,\gamma\,\frac{\vec{v}}{\left[\gamma^2(\vec{x}_\parallel - \vec{v}t)^2 + \vec{x}_\perp^2\right]^{1/2}}$$

$$= \vec{\beta}\,\frac{1}{c}\,\phi'(t,\vec{x}) = \vec{v}\,\varepsilon_0\mu_0\,\phi(t,\vec{x}) \qquad (11.1.31)$$

In spherical polar coordinates have at $t=0$

$$\phi'(\vec{x}) = \underbrace{\frac{\gamma}{\left[(\gamma^2-1)\cos^2\theta + 1\right]^{1/2}}}_{\theta\text{-dependent scaling}}\;\underbrace{\frac{Q}{4\pi\varepsilon_0}\,\frac{1}{r}}_{\text{ordinary Coulomb potential}} \qquad (11.1.32)$$

$$= \underbrace{\frac{\gamma}{\left\{r^2\left[(\gamma^2-1)\cos^2\theta + 1\right]\right\}^{1/2}}}_{(r,\theta)\text{- dep. magnitude}}\;\underbrace{\frac{Q}{4\pi\varepsilon_0}}_{\text{const.}} \qquad (11.1.33)$$

The contours of constant ϕ' and also constant $\vec{A}'(\vec{x})$ are

$$r(\theta) = K / [1 + (\gamma^2 - 1)\cos^2\theta]^{1/2} \qquad (11.1.34)$$

To describe them it is easier to go back to (11.1.30), which shows that the set of $\vec{x}$ for which $\phi'(\vec{x})$ = const is given by

$$\gamma^2 \vec{x}_{\parallel}^{\,2} + x_{\perp}^2 = \text{const.} \qquad (11.1.35)$$

These are rotationally symmetric oblate ellipsoids whose principal axis in boost direction is $\frac{1}{\gamma} = \sqrt{1-\beta^2}$ that of those perpendicular to it.

Problem 2

As $T: V \to V$ is symmetric w.r.t. η, i.e.

$$\eta(v, Tw) = \eta(Tv, w) \tag{11.2.1}$$

We have the following Lemmas

Lemma 1

If v_1 and v_2 are eigenvectors corresponding to different eigenvalues λ_1, λ_2, i.e.

$$T v_{1/2} = \lambda_{1/2} v, \quad \lambda_1 \neq \lambda_2 \tag{11.2.2}$$

then $v_1 \perp v_2$ w.r.t. η

Proof

$$\eta(v_1, T v_2) = \lambda_2 \eta(v_1, v_2)$$

$$\eta(T v_1, v_2) = \lambda_1 \eta(v_1, v_2)$$

$$\Rightarrow \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \underbrace{\eta(v_1, v_2)}_{= 0} = 0 \tag{11.2.3}$$

Lemma 2

If v is an eigenvector and $w \in v^\perp := \{ u \in V : \eta(v, u) = 0 \}$ then $Tw \in v^\perp$, i.e.

$$\mathrm{Im}(T|_{v^\perp}) \subseteq v^\perp \qquad (11.2.4)$$

or: "T leaves $v^\perp$ invariant (as set)".

Proof. Let $Tv = \lambda v$ and $w \in v^\perp$, then

$$\eta(v, Tw) = \eta(Tv, w)$$
$$= \lambda \, \eta(v, w) = 0$$

$$\Rightarrow Tw \in v^\perp . \qquad \square \qquad (11.2.5)$$

Now, if $\{e_0, e_1, e_2, e_3\}$ diagonalises T, i.e.

$$\eta(e_\alpha, e_\beta) = \eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$$

and

$$T e_\alpha = \lambda_\alpha e_\alpha \quad (\text{no sum over } \alpha)$$

then, obviously, T has a timelike eigenvector, namely e_0.

Conversely if $e_0 \in V$ is timelike and $T e_0 = \lambda_0 e_0$, By Lemma 2

T maps $e_0^{\perp} \subset V$ to itself. But $e_0^{\perp}$ is a euclidean 3d-vector space (euclidean metric $= -\eta|_{e_0^{\perp}}$)

and T is still symmetric on that space. Hence we know from linear algebra that $T|_{e_0^{\perp}}$ is diagonalisable wrt. $-\eta|_{e_0^{\perp}}$, ie there exist spacelike vectors e_a $(a = 1, 2, 3)$ such that

$$-\eta(e_a, e_b) = \delta_{ab}$$

$$T(e_a) = \lambda_a e_a \qquad (11.2.6)$$

$\Rightarrow$ T is diagonalised by $\{e_0, e_1, e_2, e_3\}$ and

$$\eta(e_\alpha, e_\beta) = \eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$$

Problem 3

$$A_\mu(x) = a_\mu \cos(k_\alpha X^\alpha + \varphi_\mu) \quad (11.3.1)$$

$$\nabla^\mu A_\mu = -k^\mu a_\mu \sin(k_\alpha X^\alpha + \varphi_\mu) \quad (11.3.2)$$

$$= 0$$

$$\Box A_\mu = \nabla^\alpha \nabla_\alpha A_\mu$$

$$= k^\alpha k_\alpha (-A_\mu) = 0 \quad (11.3.3)$$

Hence

$$F_{\alpha\beta} := \nabla_\alpha A_\beta - \nabla_\beta A_\alpha \quad (11.3.4)$$

satisfies

$$\nabla_\alpha F^{\alpha\beta} = \Box A^\beta - \nabla^\beta \nabla_\alpha A^\alpha = 0 \quad (11.3.5)$$

and, of course,

$$\nabla_{[\alpha} F_{\beta\gamma]} = 0, \quad (11.3.6)$$

so $F_{\alpha\beta}$ is a vacuum Maxwell field.

$$\nabla_\alpha A_\beta(x) = -k_\alpha a_\beta \sin(k_\lambda X^\lambda + \varphi_\beta)$$

$$= -k_\alpha a_\beta [\sin(k\cdot x)\cos\varphi_\beta + \cos(k\cdot x)\sin\varphi_\beta]$$

$$= -k_\alpha a'_\beta \sin(kx) - k_\alpha a''_\beta \cos(k\cdot x) \quad (11.3.7)$$

Where

$$a'_\beta = a_\beta \cos(\varphi_\beta)$$

$$a''_\beta = a_\beta \sin(\varphi_\beta) \qquad (11.3.8)$$

Since $k^\alpha a_\alpha = 0$ also have

$$(k \cdot a') = k^\alpha a'_\alpha = 0 \qquad (11.3.9a)$$

$$(k \cdot a'') = k^\alpha a''_\beta = 0 \qquad (11.3.9b)$$

Now,

$$F^{\alpha\mu} F^\beta{}_\mu$$

$$= \Big[(k^\alpha a'^\mu - k^\mu a'^\alpha) \sin(k \cdot x)$$

$$+ (k^\alpha a''^\mu - k^\mu a''^\alpha) \cos(k x)\Big] \times$$

$$\Big[k^\beta a'_\mu - k_\mu a'^\beta) \sin(k x)$$

$$+ (k^\beta a''_\mu - k_\mu a''^\beta) \cos(k x)\Big]$$

$$= k^\alpha k^\beta (a')^2 \sin^2(k x)$$

$$+ k^\alpha k^\beta (a' \cdot a'') \sin(k x) \cos(k x)$$

$$+ k^\alpha k^\beta (a' a'') \sin(k x) \cos(k x)$$

$$+ k^\alpha k^\beta (a'')^2 \cos^2(k x) \qquad (11.3.10)$$

All other terms vanish due to $k^2 = k \cdot a' = k \cdot a'' = 0$

$$F^{\alpha\mu} F^{\beta}{}_{\mu}$$

$$= k^{\alpha} k^{\beta} \left[a' \sin(kx) + a'' \cos(kx) \right]^2 \quad (11.3.11)$$

The energy momentum tensor is $\left(-\frac{1}{\mu_0}\right) \times$ the trace-free part of that.

But due to $k \cdot k = 0$ this is already trace free, or in other words

$$F^{\mu\nu} F_{\mu\nu} = 0 . \quad (11.3.12)$$

Hence

$$T^{\alpha\beta} = -\frac{1}{\mu_0} k_\alpha k_\beta \times$$

$$\left[a' \sin(kx) + a'' \cos(kx) \right]^2 \quad (11.3.13)$$

In particular: $T^{\alpha}{}_{\beta} V^{\beta} \sim k^{\alpha}$,

i.e. the image of the endomorphism T is one-dimensional and in Span(k). Hence T does <u>not</u> have a timelike eigenvector and T is <u>not</u> diagonalizable.

Problem 4

1)

$$\mathcal{E} = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$$

$$= \frac{1}{4!} \, \mathcal{E}_{\alpha\beta\gamma\delta} \, \theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma \wedge \theta^\delta \qquad (11.4.1)$$

$$\text{If} \quad \theta'^{\alpha} = L^\alpha{}_\beta \, \theta^\beta \quad \text{then} \qquad (11.4.2)$$

$$\mathcal{E}' = \frac{1}{4!} \, \mathcal{E}_{\alpha\beta\gamma\delta} \, L^\alpha{}_\lambda L^\beta{}_\mu L^\gamma{}_\nu L^\delta{}_\sigma \, \theta^\lambda \wedge \theta^\mu \wedge \theta^\nu \wedge \theta^\sigma$$

$$= \det\{L^\mu{}_\nu\} \, \mathcal{E}$$

$$= \mathcal{E} \quad \text{iff} \quad \det\{L^\mu{}_\nu\} = 1 \qquad (11.4.3)$$

2)

$$\mathcal{E}^{\alpha\beta\gamma\delta} \, \mathcal{E}_{\lambda\mu\nu\sigma}$$

$$= - 4! \, \delta^\alpha_{[\lambda} \, \delta^\beta_\mu \, \delta^\gamma_\nu \, \delta^\delta_{\sigma]} \qquad (11.4.4)$$

Since both sides have the same symmetries and coincide in value on $\alpha = \lambda = 0$, $\beta = \mu = 1$, $\gamma = \nu = 2$, $\delta = \sigma = 3$.
Note that the space of 4-forms in 4-dimensions is one-dimensional.
Hence symmetry plus coinciding values determine the object uniquely.

— the rest is up to you! —

Problem 5

$$(n_\wedge \wedge i_n * F)_{\alpha\beta}$$

$$= n_\alpha \quad n_\lambda * F^\lambda{}_\beta - n_\beta \; n_\lambda * F^\lambda{}_\alpha \qquad (11\,5.1)$$

$\Rightarrow$

$$[* (n_\wedge \wedge i_n * F)]_{\alpha\beta}$$

$$= \varepsilon_{\alpha\beta\mu\nu} \; n^\mu \; n_\lambda * F^{\lambda\nu}$$

$$= \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} \varepsilon^{\lambda\nu\sigma\kappa} F_{\sigma\kappa} \; n^\mu n_\lambda$$

$$= \frac{1}{2} \varepsilon_{\nu\alpha\beta\mu} \varepsilon^{\nu\lambda\sigma\kappa} F_{\sigma\kappa} n^\mu n_\lambda$$

$$= -\frac{1}{2} \left(\delta^\lambda_\alpha \delta^\sigma_\beta \delta^\kappa_\mu + \delta^\lambda_\beta \delta^\sigma_\mu \delta^\kappa_\alpha + \delta^\lambda_\mu \delta^\sigma_\alpha \delta^\kappa_\beta - \delta^\lambda_\beta \delta^\sigma_\alpha \delta^\kappa_\mu - \delta^\lambda_\alpha \delta^\sigma_\mu \delta^\kappa_\beta - \delta^\lambda_\mu \delta^\sigma_\beta \delta^\kappa_\alpha \right)$$

$$F_{\sigma\kappa} n^\mu n_\lambda$$

$$= -\frac{1}{2} \Big(\underbrace{n_\alpha F_{\beta\mu} n^\mu}_{1} + \underbrace{n_\beta F_{\mu\alpha} n^\mu}_{2} + n_\mu F_{\alpha\beta} n^\mu - \underbrace{n_\beta F_{\alpha\mu} n^\mu}_{2} - \underbrace{n_\alpha F_{\mu\beta} n^\mu}_{1} - n_\mu F_{\beta\alpha} n^\mu \Big)$$

$$= - (n_\alpha F_{\beta\mu} n^\mu + n_\beta F_{\mu\alpha} n^\mu + n^2 F_{\alpha\beta})$$

$$= (n \wedge i_n F)_{\alpha\beta} - F_{\alpha\beta} \qquad (11\,5.2)$$

since $n^2 = 1$ (unit timelike)

This shows

$$F = n \wedge i_n F - *(n \wedge i_n * F) \qquad (11.5.3)$$

Note

$$F_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{c}E_1 & \frac{1}{c}E_2 & \frac{1}{c}E_3 \\ -\frac{1}{c}E_1 & 0 & -B_3 & B_2 \\ -\frac{1}{c}E_2 & B_3 & 0 & -B_1 \\ -\frac{1}{c}E_3 & -B_2 & B_1 & 0 \end{pmatrix} \qquad (11.5.4)$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -\frac{1}{c}E_1 & -\frac{1}{c}E_2 & -\frac{1}{c}E_3 \\ \frac{1}{c}E_1 & 0 & -B_3 & B_2 \\ \frac{1}{c}E_2 & B_3 & 0 & -B_1 \\ \frac{1}{c}E_3 & -B_2 & B_1 & 0 \end{pmatrix} \qquad (11.5.5)$$

$$*F^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}F_{\mu\nu}$$

$$\Rightarrow \quad *F^{01} = \varepsilon^{0123}F_{23} = -F_{23} = B_1 \qquad (11.5.6a)$$

$$*F^{12} = \varepsilon^{1203}F_{03} = \varepsilon^{0123}F_{03}$$

$$= -F_{03} = -\frac{1}{c}E_3 \qquad (11.5.6b)$$

$$\rightsquigarrow \quad *F^{\alpha\beta} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -\frac{1}{c}E_3 & \frac{1}{c}E_2 \\ -B_2 & \frac{1}{c}E_3 & 0 & -\frac{1}{c}E_1 \\ -B_3 & -\frac{1}{c}E_2 & \frac{1}{c}E_1 & 0 \end{pmatrix} \qquad (11.5.7)$$

$$(*F)_{\alpha\beta} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & -\frac{1}{c}E_3 & \frac{1}{c}E_2 \\ B_2 & \frac{1}{c}E_3 & 0 & -\frac{1}{c}E_1 \\ B_3 & -\frac{1}{c}E_2 & \frac{1}{c}E_1 & 0 \end{pmatrix} \quad (11.5.8)$$

$$\rightsquigarrow \quad {}^*F_{\alpha\beta} {}^*F^{\alpha\beta} = 2\left(\vec{B}^2 - \frac{1}{c}\vec{E}^2\right)$$

$$F_{\alpha\beta} {}^*F^{\alpha\beta} = \frac{1}{c} 4 \vec{E} \cdot \vec{B} \quad (11.5.9)$$

$$n^\alpha = \begin{pmatrix} 1 \\ \vec{c} \end{pmatrix},$$

$$\Rightarrow \quad n^\alpha F_{\alpha b} = \frac{1}{c} E_b \quad (11.5.10)$$

$$n^\alpha {}^*F_{\alpha b} = -B_b \quad (11.5.11)$$

$$(n \wedge i_n F) = \vec{E}\text{-field alone} \quad (11.5.12)$$

$$*(n \wedge i_n {*F}) = -B\text{-field alone} \quad (11.5.13)$$

Hence

$$F = \underbrace{n_\flat \wedge i_n F}_{\frac{1}{c}\vec{E}\text{-comp.}} \underbrace{- *(n_\flat \wedge i_n F)}_{\vec{B}\text{-comp}} \quad (11.5.14)$$

= decomposition into electric and magnetic parts with respect to state of motion n.

Problem 6

$$\det(F - \lambda\, \mathrm{id}_V)$$

$$= -\frac{1}{4!}\, \varepsilon^{\alpha\beta\gamma\delta}\, \varepsilon_{\lambda\mu\nu\sigma} \left(F^{\lambda}{}_{\alpha} - \lambda \delta^{\lambda}{}_{\alpha} \right) \times$$

$$\left(F^{\mu}{}_{\beta} - \lambda \delta^{\mu}{}_{\beta} \right) \left(F^{\nu}{}_{\gamma} - \lambda \delta^{\nu}{}_{\gamma} \right) \left(F^{\sigma}{}_{\delta} - \lambda \delta^{\sigma}{}_{\delta} \right) \quad (11.6.1)$$

- The coefficients of λ^4 and λ^0 are obvious: $\det(\mathrm{id}_V) = 1$ and $\det(F)$, respectively

- The coefficients of λ^1 and λ^3 are zero: This is because F is an anti-symmetric map. To see this in an elementary fashion using matrices, recall that $F^{\alpha}{}_{\beta} = \eta^{\alpha\lambda} F_{\lambda\beta}$ and $\delta^{\alpha}{}_{\beta} = \eta^{\alpha\lambda} \eta_{\lambda\beta}$. Hence denoting the matrix $F_{\alpha\beta}$ by $\underline{F}$, and $\eta^{\alpha\beta}$ by η^{-1} we have

$$\det(F - \lambda\, \mathrm{id}_V) = \det\left[\eta^{-1} (\underline{F} - \lambda \eta)\right]$$

$$\overset{(1)}{=} -1 \det(\underline{F} - \lambda \eta)$$

$$\overset{(2)}{=} -\det(-F - \lambda \eta)$$

$$\overset{(3)}{=} -\det(F + \lambda \eta) = \det(F + \lambda\, \mathrm{id}_V) \quad (11.6.2)$$

Here we used in step (1) that $\det(\eta^{-1}) = -1$, in step (2) that the determinant of a transposed matrix equals that of the original one and that $(\underline{F} - \lambda\eta)^T = (-\underline{F} - \lambda\eta)$, and in the third step that in even dimensions $\det(X) = \det(-X)$.

This proves that $\det(F - \lambda \mathrm{id}_V) = \det(F + \lambda \mathrm{id}_V)$, which means that the characteristic polynomial has <u>no</u> odd powers in λ.

- Hence the only term that remains to be determined is the coefficient of λ^2. This term we determine from (11.6.1). There are $\binom{4}{2} = 3! = 6$ terms $\sim \lambda^2$ of the same structure and the same value:

$$-\frac{1}{4!}\, 3!\, \underbrace{\varepsilon^{\alpha\beta\gamma\delta}\, \varepsilon_{\alpha\beta\nu\sigma}}_{-2\,(\delta^\gamma{}_\nu \delta^\delta{}_\sigma - \delta^\gamma{}_\sigma \delta^\delta{}_\nu)}\, F^\nu{}_\gamma F^\sigma{}_\delta$$

$$= \frac{1}{2}\,(\underbrace{F^\nu{}_\nu F^\sigma{}_\sigma}_{=0} - F^\nu{}_\gamma F^\gamma{}_\nu)$$

$$= -\frac{1}{2}\,\mathrm{Trace}(F^2) \qquad (11.6.3)$$

Hence, in total, we get

$$P_F(\lambda) = \lambda^4 - \frac{1}{2}\lambda^2\,\mathrm{Trace}(F^2) + \det(F) \qquad (11.6.4)$$

2.)

In order to prove (12), i.e.

$$FG - (*G)(*F) = \frac{1}{2}\,\mathrm{Trace}(FG)\,\mathrm{id}_V\,,$$

we calculate the term with two $*$'s in it:

$$[(*G)(*F)]^{\alpha}{}_{\beta} = (*G)^{\alpha}{}_{\mu}(*F)^{\mu}{}_{\beta}$$

$$= (*G)^{\alpha\mu}(*F)_{\mu\beta}$$

$$= \frac{1}{2}\,\varepsilon^{\alpha\mu\gamma\delta}G_{\gamma\delta}\;\frac{1}{2}\,\varepsilon_{\mu\beta\lambda\sigma}F^{\lambda\sigma}$$

$$= -\frac{1}{4}\,\varepsilon^{\mu\alpha\gamma\delta}\varepsilon_{\mu\beta\lambda\sigma}\,G_{\gamma\delta}F^{\lambda\sigma}$$

$$= \frac{1}{4}\Big[\underbrace{\delta^{\alpha}_{\beta}\delta^{\gamma}_{\lambda}\delta^{\delta}_{\sigma}}_{1} + \underbrace{\delta^{\alpha}_{\lambda}\delta^{\gamma}_{\sigma}\delta^{\delta}_{\beta}}_{2} + \underbrace{\delta^{\alpha}_{\sigma}\delta^{\gamma}_{\beta}\delta^{\delta}_{\lambda}}_{3}$$

$$- \underbrace{\delta^{\alpha}_{\lambda}\delta^{\gamma}_{\beta}\delta^{\delta}_{\sigma}}_{4} - \underbrace{\delta^{\alpha}_{\beta}\delta^{\gamma}_{\sigma}\delta^{\delta}_{\lambda}}_{1} - \underbrace{\delta^{\alpha}_{\sigma}\delta^{\gamma}_{\lambda}\delta^{\delta}_{\beta}}_{5}\Big]$$

$$\times\, G_{\gamma\delta}F^{\lambda\sigma}$$

$$= \frac{1}{4}\Big[2\,\delta^{\alpha}_{\beta}\,G_{\mu\nu}F^{\mu\nu} + F^{\alpha\gamma}G_{\gamma\beta}$$

$$+ G_{\beta\lambda}F^{\lambda\alpha} - G_{\beta\sigma}F^{\alpha\sigma} - G_{\lambda\beta}F^{\lambda\alpha}\Big]$$

$$= \frac{1}{2}\,\delta^{\alpha}_{\beta}\,(F_{\mu\nu}G^{\mu\nu}) + F^{\alpha}{}_{\gamma}G^{\gamma}{}_{\beta}$$

$$= \Big[-\frac{1}{2}\,\mathrm{Trace}(FG)\,\mathrm{id}_V + FG\Big]^{\alpha}{}_{\beta} \qquad (11.6.5)$$

Hence

$$F G - (* G)(* F) = \frac{1}{2} \,\mathrm{Trace}\,(F G)\ \mathrm{id}_V \tag{11.6.6}$$

- Setting $G = F$ gives

$$F^2 - (* F)^2 = \frac{1}{2} \,\mathrm{Trace}\,(F^2)\ \mathrm{id}_V$$
$$= 2 I_1\ \mathrm{id}_V . \tag{11.6.7}$$

- Setting $G = * F$ gives using $* * F = - F$,

$$F(* F) = \frac{1}{4} \,\mathrm{Trace}\,(F * F)\ \mathrm{id}_V$$
$$= I_2\ \mathrm{id}_V \tag{11.6.8}$$

- If we multiply (11.6.7) with F^2 we get for the second term on the left-hand side

$$F^2 (* F)^2 = F \underbrace{F (* F)}_{I_2\ \mathrm{id}_V} (* F)$$
$$= I_2 (F * F) = I_2^2\ \mathrm{id}_V .$$

Hence

$$F^4 - 2 I_1 F^2 - I_2^2\ \mathrm{id}_V = 0 \tag{11.6.9}$$

3) Consider the tensor

$$T = F^2 - \frac{1}{4}\,\mathrm{Trace}(F^2)\,\mathrm{id}_V \qquad (11.6.10)$$

Up to a multiplicative constant $\frac{1}{\mu_0}$ this is just the energy-momentum tensor of the electromagnetic field.

Now,

$$T = F^2 - I_1\,\mathrm{id}_V \qquad (11.6.11)$$

and from (11.6.7):

$$I_1\,\mathrm{id}_V = \frac{1}{2}\left(F^2 - (*F)^2\right)$$

$$\Rightarrow \quad T = \frac{1}{2}\left(F^2 + (*F)^2\right) \qquad (11.6.12)$$

If we take the square of (11.6.11) we get

$$T^2 = \underbrace{F^4 - 2F^2 I_1}_{= I_2^2\,\mathrm{id}_V \text{ according to } (11.6.9)} + I_1^2\,\mathrm{id}_V$$

$$= \left(I_1^2 + I_2^2\right)\mathrm{id}_V \qquad (11.6.13)$$

(Rainich-Identity)

4) Since T can be written as a sum of squares according to (11.6.12) it is clearly invariant under

$$\begin{pmatrix} F \\ *F \end{pmatrix} \rightarrow \begin{pmatrix} F_\theta \\ *F_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F \\ *F \end{pmatrix} \qquad (11.6.14)$$

or

$$F_\theta = \cos\theta \, F + \sin\theta \, *F$$
$$*F_\theta = -\sin\theta \, F + \cos\theta \, *F$$

(Note that $*F_\theta = *(F_\theta)$).

Another way to see this immediately is to use the complex field strength

$$W := F + i * F \qquad (11.6.15)$$

Then

$$W \bar{W} = (F + i * F)(F - i * F)$$
$$= F^2 + (*F)^2 + i(*F F - F * F) \qquad (11.6.16)$$

However from (11.6.8) it follows from replacing F by $*F$ that

$$-(*F) F = \frac{1}{4} \text{Trace}(-*F F) \, id \, V$$
$$= -\frac{1}{4} \text{Trace}(*F F)$$
$$= -\frac{1}{4} \text{Trace}(F * F)$$
$$= - F (*F)$$

that is, F and $*F$ commute:

$$F * F = * F F . \tag{11.6.17}$$

Therefore

$$\omega \bar{\omega} = F^2 + (* F)^2$$

$$\rightarrow \quad T = \frac{1}{2} \omega \bar{\omega} \tag{11.6.18}$$

It is then obvious that T is invariant under

$$\omega \rightarrow \omega' := e^{-i\Theta} \omega \tag{11.6.19}$$

which corresponds to

$$\omega' = (\cos\Theta - i \sin\Theta)(F + i * F)$$

$$= \cos\Theta \, F + \sin\Theta * F$$
$$+ i (\cos\Theta * F - \sin\Theta \, F) .$$

i.e.

$$F \rightarrow F_\Theta := \cos\Theta \, F + \sin\Theta * F$$
$$*F \rightarrow *F_\Theta = \cos\Theta * F - \sin\Theta \, F$$

as before. In terms of $\vec{E}$ and $\vec{B}$ this is [compare (11.5.4) and (11.5.8)]

$$\vec{E} \mapsto \vec{E}_\Theta := \cos\Theta \, \vec{E} - \sin\Theta \, (c\vec{B})$$
$$B \mapsto \vec{B}_\Theta := \cos\Theta \, \vec{B} + \sin\Theta \, (\vec{E}/c) \tag{11.6.20}$$

5)

The Cayley-Hamilton theorem states that any linear map satisfies its own characteristic equation: $P_F(F) = 0$. In our case this implies an identity that we get from (11.6.4) by replacing λ with F:

$$F^4 - \tfrac{1}{2} F^2 \operatorname{Trace}(F^2) + \det F \, \mathrm{id}_V = 0 \qquad (11.6.21)$$

Comparison with (11.6.9) and recalling that $I_1 = \tfrac{1}{4}\operatorname{Trace}(F^2)$, immediately gives

$$\det(F) = -I_2^2 \qquad (11.6.22)$$

Taking the determinant of (11.6.8) we get

$$\det(F)\det(*F) = \det(I_2 \, \mathrm{id}_V) = I_2^4 \qquad (11.6.23)$$

(since we are in 4 dimensions).
Comparison of (11.6.22) and (11.6.23) gives

$$\det(*F) = -I_2^2$$

6.)

The eigenvalues of λ of F satisfy the characteristic polynomial

$$P_F(\lambda) = \lambda^4 - \frac{1}{2}\lambda^2 \, \mathrm{Trace}(F^2) + \det(F)$$
$$= 0 \qquad (11.6.24)$$

Using $\mathrm{Trace}(F^2) = 4 I_1$ and (11.6.22) this reads

$$\lambda^4 - 2 I_1 \lambda^2 - I_2^2 = 0 \qquad (11.6.25)$$

giving

$$\lambda^2 = I_1 \pm (I_1^2 + I_2^2)^{1/2} \qquad (11.6.26)$$

Which shows that for $I_1^2 + I_2^2 > 0$, i.e. if at least one of the invariants is non-zero, there are two real eigenvalues

$$\lambda_{1,2} = \pm \left[I_1 + (I_1^2 + I_2^2)^{1/2} \right]^{1/2} \qquad (11.6.27)$$

7)

From (11.6.7–9) and the additional (11.6.17) we have the following formulae:

$$(*F)^2 = F^2 - 2 I_1 \, \mathrm{id}_V \qquad (11.6.28)$$

$$(F * F) = (* F) F = I_2 \, \mathrm{id} \qquad (11.6.29)$$

$$F^4 = 2 I_1 F^2 + I_2^2 \, \mathrm{id}_V \qquad (11.6.30)$$

The first two imply that any occurrence of $(*F)^2$ and $(*F)F$ or $F(*F)$ can be eliminated in favour of F and $I_{1,2}$ and the third implies that any occurrence of F in powers 4 and above can be eliminated in favour of F^2 and $I_{1,2}$. This means that any polynomial in F and $*F$ can be written as a polynomial of F alone, unless it just contains a single $*F$, and F at most to third power.
Taking the trace a single $*F$ disappears and so does any odd power of F, in particular the third. For if we again write $F^\alpha{}_\beta = \eta^{\alpha\lambda} F_{\lambda\beta}$ or $F = \eta^{-1} \underline{F}$, then $F^3 = \eta^{-1} \underline{F} \eta^{-1} \underline{F} \eta^{-1} \underline{F}$

and

$$\mathrm{Trace}(F^3) = \mathrm{Trace}(\eta^{-1}\underline{F}\,\eta^{-1}F\,\eta^{-1}F)$$

$$= \mathrm{Trace}([\eta^{-1}\underline{F}\,\eta^{-1}\underline{F}\,\eta^{-1}\underline{F}]^T)$$

$$= \mathrm{Trace}(\underbrace{\underset{\downarrow}{\underset{-\underline{F}}{\underline{F}^T}}\,\eta^{-1}\,\underset{\downarrow}{\underset{-\underline{F}}{\underline{F}^T}}\,\eta^{-1}\,\underset{\downarrow}{\underset{-\underline{F}}{\underline{F}^T}}}_{\text{odd number of } (-) \text{ signs}}\,\eta^{-1})$$

$$= -\mathrm{Trace}(\underline{F}\,\eta^{-1}\underline{F}\,\eta^{-1}\underline{F}\,\eta^{-1})$$

cyclic property of Trace

$$= -\mathrm{Trace}(\eta^{-1}\underline{F}\,\eta^{-1}\underline{F}\,\eta^{-1}\underline{F})$$

$$= -\mathrm{Trace}(F^3)$$

$$\Rightarrow \mathrm{Trace}(F^3) = 0 \qquad (11.6.31)$$

As a result: The trace of a polynomial in F and $*F$ can be written as polynomial in I_1, I_2 and $\mathrm{Trace}(F^2) = 4I_1$; in other words: as polynomial in I_1 and I_2.