Sheet 12: Solutions

Problem 1

- $$j^\alpha(x) = qc \int \delta^{(4)}(x - z(\lambda))\, z'^\alpha(\lambda)\, d\lambda \qquad (12.1.1)$$

Here it does not matter whether we take τ or any other parameter, since

$$z'^\alpha(\lambda)\, d\lambda = \dot{z}^\alpha(\tau)\, d\tau \qquad (12.1.2)$$

One way to justify (12.1.1) is to write

$$\delta^{(4)}(x - z(\lambda)) = \delta^{(3)}(\vec{x} - \vec{z}(\lambda))\, \delta(x^0 - z^0(\lambda))$$

and

$$\delta(x^0 - z^0(\lambda)) = \frac{\delta(\lambda - \lambda_*)}{|z'^0(\lambda)|} = \frac{\delta(\lambda - \lambda_*)}{dz^0/d\lambda} \qquad (12.1.3)$$

(since $z'^0 > 0$),

where $\lambda = \lambda_*$ is the value at which $z^0(\lambda_*) = x^0$ (for timelike $z(\lambda)$ there is precisely one such value). Using (3) we can do the λ-integration in (1) and get

$$j^\alpha(x) = qc\, \delta^{(3)}(\vec{x} - \vec{z}(\lambda_*)) \left. \frac{dz^\alpha/d\lambda}{dz^0/d\lambda} \right|_{\lambda=\lambda_*} \qquad (12.1.4)$$

But $(dz^\alpha/d\lambda)/dz^0/d\lambda = (1, \vec{\beta})$

where $$\vec{\beta} = d\vec{z}/dx^0 = \frac{1}{c}\, d\vec{z}/dt = \frac{\vec{v}}{c} \qquad (12.1.5)$$

Hence using $t = x^0/c$ as parameter for $\vec{z}$ we have

$$J^0(t,\vec{x}) = q\,c\;\delta^{(3)}(\vec{x}-\vec{z}(t)) = c\,\rho(t,\vec{x}) \qquad (12.1.6)$$

$$\vec{J}(t,\vec{x}) = q\,\vec{v}\;\delta^{(3)}(\vec{x}-\vec{z}(t)) = \vec{v}\,\rho(t,\vec{x}) \qquad (12.1.7)$$

where $\rho(t,\vec{x})$ is the obvious distribution for the charge-density of a point in $\vec{x}$-space.

To show that $\nabla_\alpha J^\alpha = 0$ as a distribution we have to show that

$$\int d^4x\; f(x)\;\nabla_\alpha J^\alpha(x) = 0 \qquad (12.1.8)$$

for any test function f of - say - compact support (rapid fall-off would be sufficient). Now,

$$\int d^4x\; f(x)\;\nabla_\alpha J^\alpha(x)$$

$$= q\,c \int d\lambda\; z'^\alpha(\lambda) \int d^4x\; f(x)\;\nabla_\alpha\,\delta^{(4)}(x-z(\lambda))$$

$$= -q\,c \int d\lambda\; z'^\alpha(\lambda)\;\nabla_\alpha f(z(\lambda))$$

$$= -q\,c \int d\lambda\; \frac{d}{d\lambda} f(z(\lambda)) = -q\,c\; f(z(\lambda))\Big|_{-\infty}^{\infty}$$

$$= 0 \qquad \text{due to } f(\pm\infty) = 0. \qquad (12.1.9)$$

- $$T^{\alpha\beta}(x) = mc \int \delta^{(4)}(x - z(\tau))\, \dot{z}^\alpha(\tau)\, \dot{z}^\beta(\tau)\, d\tau \qquad (12.1.10)$$

This time we are not free to replace τ with any λ, since

$$\dot{z}^\alpha(\tau)\, \dot{z}^\beta(\tau)\, d\tau = \left(\frac{d\tau}{d\lambda}\right)^{-1} z'^\alpha(\lambda)\, z'^\alpha(\lambda)\, d\lambda \, . \qquad (12.1.11)$$

Note that for $\lambda = t = x^0/c$ this factor $(d\tau/dt)^{-1}$ is just the γ-factor. This is sometimes expressed by saying that, in contrast to electric charge q, inertial mass m is not "a relativistic invariant" but acquires a γ-factor. This is bad old-fashioned language. The origin of γ does not reside in m but in the fact that Energy - in contrast to charge - is not a scalar quantity.

Anyway, for the justification of (10) we proceed as before, using (3). Instead of (4) we here get

$$T^{\alpha\beta}(x) = mc\, \delta^{(3)}(\vec{x} - \vec{z}(\tau_*))\, \dot{z}^\alpha(\tau_*)\, \frac{\dot{z}^\beta(\tau_*)}{\dot{z}^0(\tau_*)}$$

$$= mc \left(\frac{d\tau}{dx^0}\right)^{-1} \delta^{(3)}(\vec{x} - \vec{z}(\tau_*))\, z'^\alpha(x^0)\, z'^\beta(x^0)$$

$$= mc^2 \gamma\, \delta^{(3)}(\vec{x} - \vec{z}(t))\, (1, \vec{\beta})^\alpha \otimes (1, \vec{\beta})^\beta, \qquad (12.1.12)$$

where we used again $t = x^0/c$ as parameter for $\vec{z}$. Hence we get for the components:

- Energy density

$$= T^{00}(t,\vec{x}) = mc^2 \gamma\, \delta^{(3)}(\vec{x} - \vec{z}(t)) \qquad (12.1.13)$$

- Momentum density

$$= \frac{1}{c} T^{0b}(t,\vec{x}) = m\gamma\, v^b(t)\, \delta^{(3)}(\vec{x} - \vec{z}(t)), \qquad (12.1.14)$$

- Energy current density

$$= c\, T^{b0}(t,\vec{x}) = mc^2 \gamma\, v^b(t)\, \delta^{(3)}(x - \vec{z}(t)), \qquad (12.1.15)$$

- Momentum current density

$$= T^{ab}(t,\vec{x}) = m\gamma\, v^a(t)\, v^b(t)\, \delta^{(3)}(\vec{x} - \vec{z}(t)). \qquad (12.1.16)$$

Note that $\vec{v} = d\vec{z}/dt$ and

$$\gamma = (1 - \vec{v}^2/c^2)^{-1/2}. \qquad (12.1.17)$$

Finally, in order to compute $\nabla_\alpha T^{\alpha\beta}$ as distribution we again take a test-function f and integrate

$$\int d^4x\, f(x)\, \nabla_\alpha T^{\alpha\beta}(x)$$

$$= mc \int d\tau\, \dot{z}^\alpha(\tau)\, \dot{z}^\beta(\tau) \int d^4x\, f(x)\, \nabla_\alpha \delta^{(4)}(x - z(\tau))$$

$$= -mc \int d\tau\, \dot{z}^\beta(\tau)\, \dot{z}^\alpha(\tau)\, \nabla_\alpha f(z(\tau)) \qquad (12.1.18)$$

But
$$\dot{z}^\alpha(\tau)\nabla_\alpha f(z(\tau)) = \frac{d}{d\tau} f(z(\tau)). \qquad (12.1.19)$$

Hence

$$\int d^4x\, f(x)\nabla_\alpha T^{\alpha\beta}(x) =$$
$$= -mc\int d\tau\, \dot{z}^\beta \frac{d}{d\tau} f(z(\tau))$$
$$= mc\int d\tau\, f(z(\tau))\ddot{z}^\beta(\tau)$$
$$-mc\,\underbrace{z^\beta(\tau) f(z(\tau))\Big|_{-\infty}^{\infty}}_{=0 \text{ for } f = \text{test funct.}} \qquad (12.1.20)$$

therefore

$$\int d^4x\, f(x)\nabla_\alpha T^{\alpha\beta}(x)$$
$$= mc\int d\tau\, f(z(\tau))\ddot{z}^\beta(\tau). \qquad (12.1.21)$$

For this to be zero for all f of – say – compact support we must have $\ddot{z}^\beta = 0$ i.e. $\tau \to z(\tau) \in M$ must be a straight line (= inertial motion).

Problem 2

1)

$$S_f[A] = \int d^4x \ \frac{1}{c} \left\{ -\frac{1}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta} - A_\alpha J^\alpha \right\}, \qquad (12.2.1)$$

where $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha$. (12.2.2)

The Euler-Lagrange-Equations are

$$\nabla_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\nabla_\alpha A_\beta)} \right) - \frac{\partial \mathcal{L}}{\partial A_\beta} = 0 \qquad (12.2.3)$$

where $\mathcal{L}$ is the integrand of S_f without the $\frac{1}{c}$, ie the integrand of $d^3x\, dt$:

$$S_f = \int d^3x \, dt \, \mathcal{L} \qquad (12.2.4)$$

$$\curvearrowright \quad \mathcal{L} = -\frac{1}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta} - A_\alpha J^\alpha \qquad (12.2.5)$$

Note that $\nabla_\alpha A_\beta$ appears in $F_{\alpha\beta}$ as well as $F_{\beta\alpha}$, and then in index raised form in $F^{\alpha\beta}$ and $F^{\beta\alpha}$. Hence we have four occurences of $\nabla_\alpha A_\beta$ in $F_{\alpha\beta} F^{\alpha\beta}$

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial(\nabla_\alpha A_\beta)} = -\frac{1}{\mu_0} F^{\alpha\beta} \qquad (12.2.6)$$

Clearly

$$\frac{\partial \mathcal{L}}{\partial A_\beta} = - J^\beta \qquad (12.2.7)$$

hence (3) is equivalent to

$$- \frac{1}{\mu_0} \nabla_\alpha F^{\alpha\beta} + J^\beta = 0$$

or $$\nabla_\alpha F^{\alpha\beta} = \mu_0 J^\beta \, . \qquad (12.2.8)$$

2) Given the splitting of $F_{\alpha\beta}$ into $\vec{E}$ and $\vec{B}$ field and $A_\alpha = (\frac{\phi}{c}, -\vec{A})$, $J^\alpha = (c\rho, \vec{J})$, S_f reads

$$S_f = \int dx^0 \wedge d^3x \; \frac{1}{c} \left\{ \frac{1}{2\mu_0} \frac{1}{c^2} (\vec{E}^2 - \vec{B}^2) - \rho \phi + \vec{J} \cdot \vec{A} \right\}$$

$$= \int dt \int d^3x \left\{ \frac{\varepsilon_0}{2} \vec{E}^2 - \frac{1}{2\mu_0} \vec{B}^2 \underbrace{- \rho \phi}_{} + \vec{J} \cdot \vec{A} \right\} \qquad (12.2.9)$$

energy-density of electric field

- potential energy of charge distribution ρ

$= \int dt \, \{$ electric field energy

− electric potential energy

+ magnetic parts $\}$

So at least for the electric part this is reminiscent of $L = T - V$ in mechanics, where T = kinetic energy, V = potential energy. This – in mechanics – gives the right normalization.

If we further express $\vec{E}$ and $\vec{B}$ in terms of the potentials

$$\vec{E} = -\vec{\nabla}\phi - \dot{\vec{A}} \tag{12.2.10}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{12.2.11}$$

we get with

$$S_f = \int dt \int d^3x\, \mathcal{L} =: \int dt\, L$$

$$L := \int d^3x\, \mathcal{L} = \text{Lagrangian}$$

$$= \int d^3x \left\{ \frac{\varepsilon_0}{2}\left(\dot{\vec{A}} + \vec{\nabla}\phi\right)^2 - \frac{1}{2\mu_0}\left(\vec{\nabla}\times\vec{A}\right)^2 - \rho\phi + \vec{j}\cdot\vec{A} \right\} \tag{12.2.12}$$

Conjugate momenta

$$\Pi_\phi = \frac{\delta L}{\delta \dot{\phi}} = 0 \qquad \text{(primary constraint)} \tag{12.2.13}$$

$$\vec{\Pi}_{\vec{A}} = \frac{\delta L}{\delta \dot{\vec{A}}} = \varepsilon_0\left(\dot{\vec{A}} + \vec{\nabla}\phi\right) \tag{12.2.14}$$

$\Rightarrow$ Hamiltonian

$$H = \int d^3x \left(\Pi_\phi \dot{\phi} + \vec{\Pi}_{\vec{A}} \dot{\vec{A}} - \mathcal{L} \right) \tag{12.2.15}$$

$\Rightarrow$

$$H = \int d^3x \left\{ \vec{\Pi}_A \left(\frac{1}{\varepsilon_0} \vec{\Pi}_A - \vec{\nabla}\phi \right) - \frac{\varepsilon_0}{2} \vec{\Pi}_A^2 + \frac{1}{2\mu_0} (\vec{\nabla} \times \vec{A})^2 + \rho\phi - \vec{j}\cdot\vec{A} \right\}$$

$$= \int_\Sigma d^3x \left\{ \frac{1}{2\varepsilon_0} \vec{\Pi}_A^2 + \phi(\rho + \vec{\nabla}\cdot\vec{\Pi}_A) + \frac{1}{2\mu_0}(\vec{\nabla}\times\vec{A})^2 - \vec{j}\cdot\vec{A} \right\}$$

$$- \int_{\partial\Sigma} d\sigma\, (\vec{n}\cdot\vec{\Pi}_A)\,\phi \qquad (12.2.16)$$

The right sign and normalisation appears from having $\frac{1}{2}\Pi^2$ (up to ε_0) as integrand in Hamiltonian. Also, the magnetic field energy $\sim \int d^3x \frac{1}{2\mu_0}(\vec{\nabla}\times\vec{A})$ now enters with right sign and normalisation. Note that since no time-derivative $\dot{\phi}$ of ϕ enters there is no Π_ϕ; variation of H with respect to ϕ just gives

$$\frac{\delta H}{\delta\phi} = (\rho + \vec{\nabla}\Pi) = \rho - \varepsilon\vec{\nabla}\vec{E} = 0$$

$$\Leftrightarrow \quad \vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho \qquad (12.2.17)$$

Maxwell's equations are now equivalent to Hamilton's equations

$$\dot{\vec{A}} = \frac{\delta H}{\delta \vec{\Pi}_A} \qquad (12.2.18)$$

$$\dot{\vec{\Pi}}_A = - \frac{\delta H}{\delta \vec{A}} \qquad (12.2.19)$$

and also

$$\underbrace{0 = \dot{\Pi}_\phi = - \frac{\delta H}{\delta \phi}}_{\Leftrightarrow \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0} \quad \text{(secondary constraint)} \qquad (12.2.20)$$

3) $$S_p[Z] = - mc \int d\lambda \left[Z'_\alpha(\lambda) Z'^\alpha(\lambda) \right]^{1/2} \qquad (12.2.21)$$

= action of free particle

For charged particle have current

$$J^\alpha(x) = q\,c \int d\lambda\, \delta^{(4)}(x - Z(\lambda))\, Z'^\alpha(\lambda) \qquad (12.2.22)$$

and interaction

$$S_{int} = \int d^4x\, \frac{1}{c} \left\{ - A_\alpha(x) J^\alpha(x) \right\}$$

$$= - q \int d\lambda\, Z'^\alpha(\lambda) \int d^4x\, A_\alpha(x)\, \delta^{(4)}(x - Z(\lambda))$$

$$= - q \int d\lambda\, A_\alpha(Z(\lambda))\, Z'^\alpha(\lambda) \qquad (12.2.23)$$

Hence

$$S_p + S_{int}$$

$$= \int d\lambda \left\{ - mc \left(z'_\alpha(\lambda) z'^\alpha(\lambda) \right)^{1/2} - q \, A_\alpha(z(\lambda)) z'^\alpha(\lambda) \right\}$$

$$=: \int d\lambda \; L(z(\lambda), z'(\lambda)) \tag{12.2.24}$$

The Euler-Lagrange-Equations are

$$\frac{d}{d\lambda} \left(\frac{\partial L}{\partial z'^\alpha} \right) - \frac{\partial L}{\partial z^\alpha} = 0 . \tag{12.2.25}$$

Now, $$\frac{\partial L}{\partial z'^\alpha} = - mc \left(z'_\beta z'^\beta \right)^{-1/2} z'^\alpha - q \, A_\alpha(z) \tag{12.2.26}$$

Since

$$c \, d\tau = \left(z'_\beta z'^\beta \right)^{1/2} d\lambda$$

or $$\frac{d}{d\tau} = c \left(z'_\beta z'^\beta \right)^{-1/2} \frac{d}{d\lambda} \tag{12.2.27}$$

this can be written as

$$\frac{\partial L}{\partial z'^\alpha} = - m \dot{z}^\alpha - q \, A_\alpha(z) \tag{12.2.28}$$

Then $$\frac{d}{d\lambda} \frac{\partial L}{\partial z'^\alpha} = - m (\dot{z}_\alpha)' - q (\nabla_\beta A_\alpha) z'^\beta \tag{12.2.29}$$

and
$$\frac{\partial L}{\partial Z^\alpha} = -q \nabla_\alpha A_\beta Z'^\beta \tag{12.2.30}$$

The EL-Equations (25) are therefore equivalent to

$$-m(\dot{Z}_\alpha)' - q\,(\nabla_\beta A_\alpha - \nabla_\alpha A_\beta)\,Z'^\beta = 0$$

$$\Leftrightarrow \; -m\ddot{Z}_\alpha - q\,(\nabla_\beta A_\alpha - \nabla_\alpha A_\beta)\,\dot{Z}^\beta = 0$$

$$\Leftrightarrow \; m\ddot{Z}^\alpha = q\,F^\alpha{}_\beta\,\dot{Z}^\beta \tag{12.2.31}$$

which is just Lorentz-force law.

Problem 3

Abraham Lorentz equation for a charged particle in an electromagnetic field with radiation-reaction reads

$$m \ddot{\vec{z}} = q(\vec{E} + \dot{\vec{z}} \times \vec{B}) + \sigma m \dddot{\vec{z}} \qquad (12.3.1)$$

which is supposed to be valid for

$$\|\dot{\vec{z}}\| / c \ll 1 . \qquad (12.3.2)$$

The constant σ is

$$\sigma = \frac{\mu_0 q^2}{6\pi c m} \qquad (12.3.3)$$

1.)

$$\sigma = \frac{4}{3} \cdot \frac{\mu_0 c^2 q^2}{8\pi c^3 m} = \frac{4}{3} \cdot \underbrace{\frac{q^2}{8\pi \varepsilon_0 m c^2}}_{R_q} \cdot \frac{1}{c} \qquad (12.3.4)$$

$$R_q := \frac{q^2}{8\pi \varepsilon_0 m c^2} \qquad (12.3.5)$$

is the "classical charge radius". It is the radius outside which the spherically symmetric electric Coulomb field of a charge q has an energy of mc^2.

Proof. $$E(R) = \int_{B_R \subset \mathbb{R}^3} \frac{\varepsilon_0}{2} \vec{E}^2 \, d^3x \qquad (12.3.6)$$

with $\vec{E} = \frac{q}{4\pi \varepsilon_0 r^2} \vec{n}$ get

$$E(R) = \frac{\varepsilon_0}{2} \frac{q^2}{(4\pi\varepsilon_0)^2} \int_R^\infty \frac{dr}{r^4} r^2 \underbrace{\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi}_{4\pi}$$

$$= \frac{q^2}{8\pi\varepsilon_0} \left(-\frac{1}{r}\right)\Big|_R^\infty = \frac{q^2}{8\pi\varepsilon_0 R} \qquad (12.3.7)$$

If this is to be equal to mc^2 we get

$$\frac{q^2}{8\pi\varepsilon_0 R} = mc^2$$

or $\quad R = R_q := \dfrac{q^2}{8\pi\varepsilon_0 mc^2} \qquad (12.3.8)$

σ is $\frac{4}{3}$ times the time $\frac{R_q}{c}$ that it takes light to travel a distance R_q.

Note that for a particle with charge q one cannot assume a particle size less than R_q without a negative (binding?) energy somewhere, since its mere Coulomb field already accounts for more energy than mc^2.

Consider σ for an electron. Rather than just calculating the number we wish to relate it to other quantities.

$$\lambda\!\!\!^{-}_e = \frac{\hbar}{mc} = \text{reduced Compton wave length}$$

$$\alpha = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{\hbar c} = \text{Fine structure constant}$$

Hence

$$\alpha \lambda_e = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{mc^2}$$

So that for the electron we have

$$\sigma_e = \frac{2}{3} \alpha \cdot ƛ_e \cdot \frac{1}{c} \qquad (12.3.9)$$

$$\alpha = \frac{1}{137},$$

$$ƛ_e = \frac{1}{2\pi} \lambda_e = \frac{2.43 \times 10^{-12}\,m}{2\pi}$$

$$= 3.87 \times 10^{-13}\,m$$

$$\Rightarrow \quad R_e = 1.88 \times 10^{-15}\,m \qquad (12.3.10)$$

$$\sigma_e = \frac{1}{c} R_e = 6.28 \times 10^{-24}\,s \qquad (12.3.11)$$

2) A first obvious guess for a generalisation of (1) would be to just replace all

$$\frac{d^n \vec{z}}{dt^n} \quad \text{by} \quad \frac{d^n z^\alpha}{d\tau^n} \qquad (12.3.12)$$

i.e.

$$m \ddot{z}^\alpha = q F^\alpha{}_\beta \dot{z}^\beta + \sigma m \dddot{z}^\alpha \qquad (12.3.13)$$

However, since

$$\dot{Z}_\alpha \ddot{Z}^\alpha = 0 \qquad (12.3.14)$$

and

$$\dot{Z}_\alpha F^\alpha{}_\beta \dot{Z}^\beta = F_{\alpha\beta} \dot{Z}^\alpha \dot{Z}^\beta = 0, \qquad (12.3.15)$$

this would imply

$$0 = \dot{Z}_\alpha \dddot{Z}^\alpha = (\underbrace{\dot{Z}_\alpha \ddot{Z}^\alpha}_{=0})^\bullet - \ddot{Z}_\alpha \ddot{Z}^\alpha$$

$$\Rightarrow \quad \ddot{Z}_\alpha \ddot{Z}^\alpha = 0 \qquad (12.3.16)$$

and since $\ddot{Z}$ is spacelike (follows from (14))

$$\ddot{Z}^\alpha = 0$$

Inserting this back into (13) then implies

$$F^\alpha{}_\beta \dot{Z}^\beta = 0 \qquad (12.3.17)$$

i.e. F must have a kernel containing a timelike element. This clearly implies det(F) = 0 and hence, by Problem 6 of Sheet 11 that $I_2 = \frac{1}{4}\,\mathrm{Trace}\,(F * F) = 0$ i.e. $\vec{E}\cdot\vec{B} = 0$ in any frame. More specifically, if we go to the frame defined by the timelike kernel element V, $F^\alpha{}_\beta V^\beta = 0$, so that $V = (1, \vec{0}^T)$ and

$$F^\alpha{}_\beta = \begin{pmatrix} 0 & \frac{1}{c}\vec{E} \\ \frac{1}{c}\vec{E} & \underline{B} \end{pmatrix} \qquad (12.3.18)$$

then

$$F v = \begin{pmatrix} 0 & \frac{1}{c}\vec{E}^T \\ \frac{1}{c}\vec{E} & \underline{B} \end{pmatrix} \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{c}\vec{E} \end{pmatrix} = 0$$

$$\Rightarrow \vec{E} = 0 \tag{12.3.19}$$

i.e. in that frame the field must be purely magnetic. The invariant characterisation of that is

$$\vec{B}^2 - \frac{1}{c^2}\vec{E}^2 = \frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} \geq 0 \tag{12.3.20}$$

$$\vec{B} \cdot \vec{E} = \frac{c}{4} F_{\alpha\beta} {*F}^{\alpha\beta} = 0 \tag{12.3.21}$$

or, equivalently

$$I_1 = \frac{1}{4} \mathrm{Trace}(F^2) \leq 0 \tag{12.3.22}$$

$$I_2 = \frac{1}{4} \mathrm{Trace}(F * F) = 0 \tag{12.3.23}$$

So for general fields there simply is no solution. And for those for which solutions exist $\ddot{z}^\alpha = 0$, which is physically uninteresting.

The mathematical reason for this extra condition is that in going from (1) to (13) we add one more independent condition ((1) is an equation for $\vec{z} \in \mathbb{R}^3$, (13) for $z^\alpha \in \mathbb{R}^4$), unless the extra term added with σ is also perpendicular to $\dot{z}$. So the second guess for a relativistic version of (1) is to just project $\dddot{z}$ $\perp$ to $\dot{z}$

$$P^{\perp}_{\dot{z}}(\dddot{z}) = \dddot{z} - \frac{1}{c^2}\dot{z}\,(\dot{z}\cdot\dddot{z}) \qquad (12.3.24)$$

But
$$\dot{z}\cdot\dddot{z} = \underbrace{(\dot{z}\cdot\ddot{z})^{\cdot}}_{=0} - \ddot{z}^2$$

hence

$$P^{\perp}_{\dot{z}}(\dddot{z}) = \dddot{z} + \frac{1}{c^2}\dot{z}\,(\ddot{z}\cdot\ddot{z}) \qquad (12.3.25)$$

So the second-guess equation is

$$m\ddot{z}^{\alpha} = q\,F^{\alpha}{}_{\beta}\,\dot{z}^{\beta} + \tau m\left(\dddot{z}^{\alpha} + \frac{1}{c^2}\dot{z}^{\alpha}\,\ddot{z}_{\beta}\,\ddot{z}^{\beta}\right) \qquad (12.3.26)$$

<u>Lorentz-Dirac-Equation</u>

3)
In Problem 4 of Sheet 10 we have seen that $P^{\perp}_{\dot{z}}(\dddot{z}) = 0$ is precisely the condition for <u>constant</u> <u>acceleration</u>.
Hence all solutions of

$$m\ddot{z}^{\alpha} = q\,F^{\alpha}{}_{\beta}\,\dot{z}^{\beta} \qquad (12.3.27)$$

which happen to be of constant acceleration also solve (26). This is the case for the solution found in Problem 5 of Sheet 9, but not for its generalization in Problem 2 of

Sheet 10.

4.) For $\dot{z} = c\,u$ (26) reads for $F_\alpha{}^\beta = 0$:

$$\dot{u} = \sigma\,(\ddot{u} + u\,(\dot{u}\cdot\dot{u})) \tag{12.3.28}$$

Solutions to this equation remain in the plane $\mathrm{Span}\{u, \dot{u}\}$. Since $u^2 = 1$ we may write

$$u = \cosh(\varrho)\, e_0 + \sinh(\varrho)\, e_1 \tag{12.3.29}$$

$$\dot{u} = [\sinh(\varrho)\, e_0 + \cosh(\varrho)\, e_1]\,\dot{\varrho} \tag{12.3.30}$$

$$\ddot{u} = u\,\dot{\varrho}^2 + \dot{u}\,(\ddot{\varrho}/\dot{\varrho}) \tag{12.3.31}$$

$$\leadsto \dot{u}^2 = -\dot{\varrho}^2 \tag{12.3.32}$$

Hence (28) reads

$$\dot{u} = \sigma\,[\cancel{u\,\dot{\varrho}^2} + \dot{u}\,(\ddot{\varrho}/\dot{\varrho}) - \cancel{u\,\dot{\varrho}^2}] \tag{12.3.33}$$

$$\Leftrightarrow \quad \sigma\,\ddot{\varrho}/\dot{\varrho} = 1$$

$$\Leftrightarrow \quad \ddot{\varrho} = \dot{\varrho}/\sigma \tag{12.3.34}$$

$$\Leftrightarrow \quad \dot{\varrho}(\tau) = \dot{\varrho}_0 \exp(\tau/\sigma) \tag{12.3.35}$$

$$\Leftrightarrow \quad \varrho(\tau) = \sigma\,\dot{\varrho}_0 \exp(\tau/\sigma) + \varrho_0$$

$\Rightarrow$ Exponential growth of rapidity !