Sheet 2: Solutions

Problem 1

$$T = \frac{1}{2} m_1 \dot{\vec{x}}_1^2 + \frac{1}{2} m_2 \dot{\vec{x}}_2^2 \tag{2.1.1a}$$

$$U = U(\|\vec{x}_1 - \vec{x}_2\|) \tag{2.1.1b}$$

Centre of mass coordinates

$$\vec{r} := \vec{x}_1 - \vec{x}_2 \tag{2.1.2a}$$

$$\vec{R} := (m_1 \vec{x}_1 + m_2 \vec{x}_2)/(m_1 + m_2) \tag{2.1.2b}$$

$$\vec{x}_1 = \vec{R} + \frac{m_2}{m_1+m_2} \vec{r} \tag{2.1.3a}$$

$$\vec{x}_2 = \vec{R} - \frac{m_1}{m_1+m_2} \vec{r} \tag{2.1.3b}$$

$$\curvearrowright \quad \dot{\vec{x}}_1 = \dot{\vec{R}} + \frac{m_2}{m_1+m_2} \dot{\vec{r}} \tag{2.1.4a}$$

$$\dot{\vec{x}}_2 = \dot{\vec{R}} - \frac{m_1}{m_1+m_2} \dot{\vec{r}} \tag{2.1.4b}$$

$$\curvearrowright \quad T = \frac{1}{2} m_1 \left(\dot{\vec{R}} + \frac{m_2}{m_1+m_2} \dot{\vec{r}} \right)^2 + \frac{1}{2} m_2 \left(\dot{\vec{R}} - \frac{m_1}{m_1+m_2} \dot{\vec{r}} \right)^2$$

$$= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 \tag{2.1.5}$$

where

$$M := m_1 + m_2 \tag{2.1.6a}$$

$$\mu := \frac{m_1 m_2}{m_1 + m_2} \tag{2.1.6b}$$

$$L = T - U$$

$$= \frac{1}{2} m_1 \dot{\vec{x}}_1^2 + \frac{1}{2} m_2 \dot{\vec{x}}_2 - U(\|\vec{x}_1 - \vec{x}_2\|) \qquad (2.1.7)$$

$$= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 - U(r) \qquad (2.1.8)$$

Euler-Lagrange-Equations:

$$\frac{\partial L}{\partial \dot{\vec{x}}_a} = m_a \dot{\vec{x}}_a$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{x}}_a} = m \ddot{\vec{x}}_a$$

$$\frac{\partial L}{\partial \vec{x}_1} = -U' \frac{\vec{x}_1 - \vec{x}_2}{\|\vec{x}_1 - \vec{x}_2\|}$$

$$\frac{\partial L}{\partial \vec{x}_2} = -U' \frac{\vec{x}_2 - \vec{x}_1}{\|\vec{x}_2 - \vec{x}_1\|}$$

$$\Rightarrow \quad m_1 \ddot{\vec{x}}_1 + U' \frac{\vec{x}_1 - \vec{x}_2}{\|\vec{x}_1 - \vec{x}_2\|} = 0 \qquad (2.1.9a)$$

$$m_2 \ddot{\vec{x}}_2 + U' \frac{\vec{x}_2 - \vec{x}_1}{\|\vec{x}_2 - \vec{x}_1\|} = 0 \qquad (2.1.9b)$$

$$\frac{\partial L}{\partial \dot{\vec{R}}} = M \dot{\vec{R}} \quad , \quad \frac{\partial L}{\partial \vec{R}} = 0$$

$$\frac{\partial L}{\partial \dot{\vec{r}}} = \mu \dot{\vec{r}} \quad , \quad \frac{\partial L}{\partial \vec{r}} = -U' \frac{\vec{r}}{r}$$

$$\Rightarrow M \ddot{\vec{R}} = 0 , \quad \mu \ddot{\vec{r}} + U' \frac{\vec{r}}{r} = 0 \qquad (2.1.10a,b)$$

The momenta are

$$\vec{p}_a := \frac{\partial L}{\partial \dot{\vec{x}}_a} = m_a \dot{\vec{x}}_a \qquad (2.1.11)$$

$$\vec{P} := \frac{\partial L}{\partial \dot{\vec{R}}} = M \dot{\vec{R}} \qquad (2.1.12a)$$

$$\vec{p} := \frac{\partial L}{\partial \dot{\vec{r}}} = \mu \dot{\vec{r}} \qquad (2.1.12b)$$

The Hamiltonian is

$$H = \sum_{a=1}^{2} \vec{p}_a \dot{\vec{x}}_a - L$$

$$= \sum_{a=} \frac{\vec{p}_a^{\,2}}{2m_a} + U(\|\vec{x}_1 - \vec{x}_2\|)$$

$$= \vec{P}\dot{\vec{R}} + \vec{p}\,\dot{\vec{r}} - L$$

$$= \frac{\vec{P}^2}{2M} + \frac{\vec{p}^{\,2}}{2\mu} + U(r) \qquad (2.1.13)$$

Note: $\vec{r}$ and $\vec{p}$ are conjugate, so that the conjugate variable $\vec{R}$ to $\vec{P}$ does not appear in the Hamiltonian

Under Galilei-boosts have

$$\vec{x}_a \to \vec{x}_a' = \vec{x}_a + \vec{v}t \qquad (2.1.14)$$

$$\vec{R} \to \vec{R}' = \vec{R} + \vec{v}t \qquad (2.1.15a)$$

$$\vec{r} \to \vec{r}' = \vec{r} \qquad (2.1.16b)$$

For the actively transformed trajectories the Lagrangian is

$$L' = \frac{1}{2}\sum_{a=1}^{2} m_a \dot{\vec{x}}'_a(t) - \mathcal{U}$$

$$= \frac{1}{2}\sum_{a=1}^{2} m_a \left(\dot{\vec{x}}_a(t) + \vec{v}\right) - \mathcal{U}$$

$$= L + \underbrace{\sum_{a=1}^{2} m_a \dot{\vec{x}}_a(t)}_{M\dot{\vec{R}}} \cdot v + \frac{1}{2} M \vec{v}^2 - \mathcal{U} \qquad (2.1.17)$$

or

$$L' = \frac{1}{2} M \dot{\vec{R}}'^2 + \frac{1}{2}\mu \dot{\vec{r}}'^2 - \mathcal{U}$$

$$= \frac{1}{2} M \left(\dot{\vec{R}} + \vec{v}\right)^2 + \frac{1}{2}\mu \dot{\vec{r}}^2 - \mathcal{U}$$

$$= L + M\dot{\vec{R}} \cdot \vec{v} + \frac{1}{2} M \vec{v}^2 - \mathcal{U} \qquad (2.1.18)$$

leading to the same result.

Now

$$\frac{\partial L'}{\partial \dot{\vec{x}}_a} = \vec{p}'_a = \frac{\partial L}{\partial \dot{\vec{x}}_a} + m_a \vec{v} = \vec{p}_a + m_a \vec{v} \qquad (2.1.19)$$

$$\frac{\partial L'}{\partial \dot{\vec{R}}} = \vec{P}' = \frac{\partial L}{\partial \dot{\vec{R}}} + M\vec{v} = \vec{P} + M\vec{v} \qquad (2.1.20)$$

constant

So that

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{\vec{x}}_a} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{x}}_a} \qquad (2.1.21)$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{\vec{R}}} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{R}}} \qquad (2.1.22)$$

Equations of motion for L and L' are the same. The reason is that they differ by a total time derivative

$$L'[\vec{X}_a(t)] = L[\vec{X}_a(t)]$$

$$+ \frac{d}{dt}\left\{ \sum m_a \vec{X}_a(t) \cdot \vec{V} + \frac{1}{2} M \vec{V}^2 t \right\} \qquad (2.1.23)$$

or

$$L'[\vec{R}(t), \vec{r}(t)] = L[R(t), \vec{r}(t)]$$

$$+ \frac{d}{dt}\left\{ M \vec{R} \cdot \vec{V} + \frac{1}{2} M \vec{V}^2 t \right\} \qquad (2.1.24)$$

It follows that boosts are Noether-symmetries (defined by the requirement that the Lagrangian changes at most by a total time derivative) and that the corresponding conserved quantities are

$$\left.\frac{\partial}{\partial \vec{V}}\right|_{\vec{V}=\vec{0}} \left\{ \sum_{a=1}^{2} \underbrace{\Delta \vec{X}_a}_{\vec{V} t} \underbrace{\frac{\partial L}{\partial \dot{\vec{X}}_a}}_{\vec{p}_a} - \left(M \vec{R} \vec{V} + \frac{1}{2} M \vec{V}^2 t \right) \right\}$$

$$= \left.\frac{\partial}{\partial \vec{V}}\right|_{\vec{V}=0} \left\{ \vec{V} t \sum_{a=1}^{2} \vec{p}_a - \left(M \vec{R} \vec{V} + \frac{1}{2} M \vec{V}^2 t \right) \right\}$$

$$= t \vec{P} - M \vec{R} \qquad (2.1.25)$$

where $\vec{P} := \vec{p}_1 + \vec{p}_2$ = total momentum

The Noether conserved quantity for boost symmetry is

$$\vec{Q} = t\vec{P} - M\vec{R} \qquad (2.1.25)$$

Note that $\vec{P}(t)$ and $\vec{R}(t)$ are a priori time dependent whereas $\vec{Q}$ is not (it's conserved). However, space-translation invariance gives another Noether conserved quantity, which is just $\vec{P}$. Hence we have from (2.1.25):

$$R(t) = \frac{1}{M}\left(t\vec{P} - Q\right) \qquad (2.1.26)$$

where $\vec{Q}$ and $\vec{P}$ are constant.
This says that the centre of mass moves on a straight line with constant speed $\vec{P}/M$.

Intermezzo: A little excursion to Noether's theorem.

Consider a Lagrangian

$$L: \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$$

$$(t, q^\alpha, v^\alpha) \mapsto L(t, q^\alpha, v^\alpha)$$

Here q^α, $\alpha \in \{1, \dots, N\}$ are any N coordinates and v^α the correspon-ding velocities.

Given a curve

$$\gamma : \mathbb{R} \to \mathbb{R}^N$$

$$t \mapsto \gamma^\alpha(t)$$

with velocity

$$\dot\gamma : \mathbb{R} \to \mathbb{R}^N$$

$$t \mapsto \dot\gamma^\alpha(t)$$

L becomes a function of t alone by

$$(L \circ \gamma)(t) := L(t, q^\alpha = \gamma^\alpha(t), v^\alpha = \dot\gamma^\alpha(t))$$

The "action" of the curve γ is

$$S[\gamma] = \int_{t_1}^{t_2} dt\,(L \circ \gamma)(t) \qquad (2.1.27)$$

where the range of integration $[t_1, t_2]$ is that of the curve γ.

Suppose we have a one-parameter family of curves

$$\gamma : \mathbb{R}^2 \to \mathbb{R}^N$$

$$(t, s) \mapsto \gamma_s(t) \qquad (2.1.28)$$

between the <u>same</u> endpoints, i.e. $\gamma_s(t_{1,2})$ are independent of s, then the action becomes a function of s

$$S[\gamma_s] = S(s) := \int_{t_1}^{t_2} dt\,(L \circ \gamma_s)(t) \qquad (2.1.29)$$

S is said to be "stationary" at the path $\gamma_0 := \gamma_{s=0}$ iff

$$\frac{d}{ds}\Big|_{s=0} S[\gamma_s] = 0$$

$$\Leftrightarrow \frac{d}{ds}\Big|_{s=0} \int_{t_1}^{t_2} (L \circ \gamma_s)(t)\,dt = 0$$

$$= \frac{d}{ds}\Big|_{s=0} \int_{t_1}^{t_2} L(t, q^\alpha = \gamma_s^\alpha(t), v^\alpha = \dot{q}_s^\alpha(t)) \qquad (2.1.30)$$

$$= \int_{t_1}^{t_2} dt \left\{ \left[\frac{\partial L}{\partial q^\alpha}\right] \frac{\partial \gamma_s(t)}{\partial s}\Big|_{s=0} + \left[\frac{\partial L}{\partial v^\alpha}\right] \frac{\partial^2 \gamma_s(t)}{\partial s\, \partial t}\Big|_{s=0} \right\} \qquad (2.1.31)$$

(here $[\,\cdot\cdot\,]$ denotes evaluation at $q^\alpha = \gamma_0^\alpha(t)$ and $v^\alpha = \dot{\gamma}_0^\alpha(t)$)

$$= \int_{t_1}^{t_2} dt \left\{ \left(\left[\frac{\partial L}{\partial q^\alpha}\right] - \frac{d}{dt}\left[\frac{\partial L}{\partial v^\alpha}\right] \right) \frac{\partial \gamma_s(t)}{\partial s}\Big|_{s=0} + \frac{d}{dt}\left(\left[\frac{\partial L}{\partial v^\alpha}\right] \frac{\partial \gamma_s^\alpha(t)}{\partial s}\Big|_{s=0} \right) \right\} \qquad (2.1.32)$$

first application of (2.1.32)

If $\gamma_s(t_{1,2})$ is independent of s, i.e. if the endpoints of γ are fixed, the last term does not contribute and stationarity of the action for all variations implies that

$$\odot = \int_{t_1}^{t_2} dt \left(\left[\frac{\partial L}{\partial q^\alpha}\right] - \frac{d}{dt}\left[\frac{\partial L}{\partial v^\alpha}\right] \right) f(t) = 0$$

where $f(t) := \frac{\partial \gamma_s(t)}{\partial s}\Big|_{s=0}$ (2.1.33)

can be any function of t with $f(t_{1,2}) = 0$. But that implies

$$\left[\frac{\partial L}{\partial q^\alpha}\right] - \frac{d}{dt}\left[\frac{\partial L}{\partial v^\alpha}\right] = 0 \qquad (2.1.34)$$

(Euler-Lagrange-Equation)

Note that $[\cdots] := (\cdots)\Big|_{\substack{q=\gamma_0(t)\\ v=\dot{\gamma}_0(t)}}$ (2.1.35)

and this is sometimes written as

$$\text{"}\left\{\frac{\partial L}{\partial \gamma^\alpha(t)} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\gamma}^\alpha(t)}\right) = 0\right\}\text{"} \qquad (2.1.36)$$

Suppose next that γ_0 satisfies the Euler Lagrange equation and that γ_s is a one parameter variation of γ_0 for which we have

$$\frac{\partial}{\partial s}\Big|_{s=0} L\left(t, \gamma_s^\alpha(t), \dot{\gamma}_s^\alpha(t)\right)$$

$$= \frac{d}{dt} K\left(t, \gamma_0^\alpha(t), \dot{\gamma}_0^\alpha(t)\right) \qquad (2.1.37)$$

for some function $K : \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$. Then it follows from (2.1.32) that

$$\int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial v^\alpha}\right] \frac{\partial \gamma_s(t)}{\partial s}\Big|_{s=0}$$

$$= \int_{t_1}^{t_2} dt \frac{d}{dt} K\left(t, \gamma_0^\alpha(t), \dot{\gamma}_0^\alpha(t)\right) \qquad (2.1.37)$$

second application of (2.1.32)

Since the interval $[t_1, t_2]$ of integration is arbitrary, we may conclude from this that

$$\left[\frac{\partial L}{\partial v^2}\right] \frac{\partial \gamma_s^2}{\partial s}\Big|_{s=0} - K(t, \gamma_0^2(t), \dot\gamma_0^2(t))$$

$$=: Q \tag{2.1.38}$$

is independent of t

Q is the Noether conserved quantity associated with the symmetry given by the one-parameter family $s \longmapsto \gamma_s$ of curves.

All this applies to our case of boost symmetry. We had for 2 particles

$$\gamma^2(t) = (\vec{X}_1(t), \vec{X}_2(t)) \in \mathbb{R}^6$$

$$\gamma_s^2(t) = (\vec{X}_1(t) + \vec{V}(s)\,t, \vec{X}_2(t) + \vec{V}(s)) \tag{2.1.39}$$

where $\vec{V}(s=0) = \vec{0}$. We can, for example, take

$$V(s) = s\,\vec{n} \tag{2.1.40}$$

for some fixed $\vec{n} \in \mathbb{R}^3$ with $\|\vec{n}\| = 1$. As $\vec{n}$ ranges over all directions we get all possible boosts.

Then, from (2.1.23, 24), we get

$$\frac{\partial}{\partial s}\Big|_{s=0} L[\vec{X}_a(t) + s\vec{n}t]$$

$$= \frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\left\{ s M\vec{R}\cdot\vec{n} + s^2 \frac{1}{2} M t \right\}$$

$$= \frac{d}{dt} M\vec{R}\cdot\vec{n} \qquad (2.1.40)$$

hence

$$K = M\vec{R}\cdot\vec{n} \qquad (2.1.41)$$

On the other hand, in our case, the first term on the left-hand side of (2.1.38) is

$$\left[\frac{\partial L}{\partial v^a}\right] \frac{\partial \gamma^a_s}{\partial s}\Big|_{s=0} =$$

$$(\vec{p}_1 + \vec{p}_2)\,\vec{n}t = \vec{n}\cdot\vec{P}\,t \qquad (2.1.42)$$

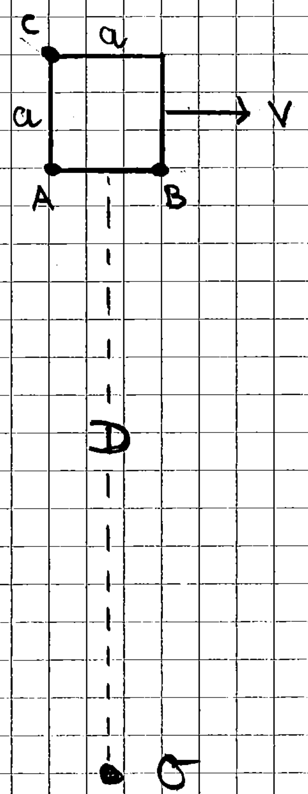
$$\curvearrowright\ Q = \vec{n}\,(\vec{P}t - M\vec{R}) \qquad (2.1.43)$$

As this is conserved for all possible boost-directions $\vec{n}$, we get the triple of Noether conserved quantities

$$\vec{Q} = (\vec{P}t - M\vec{R}), \qquad (2.1.44)$$

as stated in (2.1.25).

Problem 2



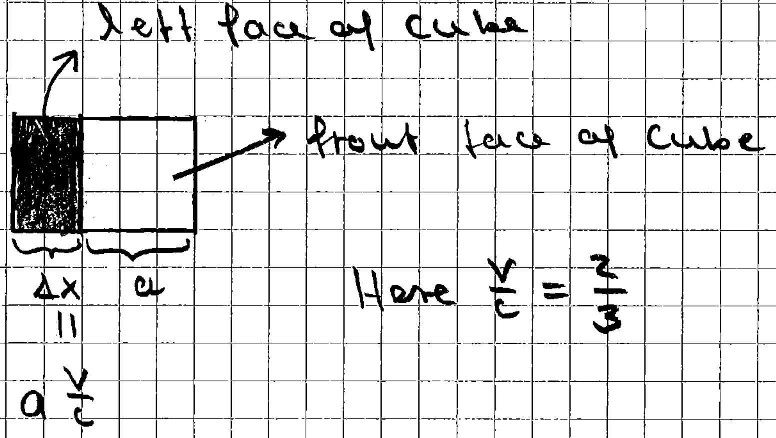
(2.2.1)

Light from the front corners A and B take time $t = D/c$, $c =$ vel. of light, to reach observer. The light from the corner C at the back that can also reach O because the cube moves "out of the way" to the right takes time $t + \Delta t$, with

$$\Delta t = \frac{a}{c} \qquad (2.2.2)$$

Therefore, in order to reach the observer at the same time as the light from A and B it has to be sent off earlier by Δt. At that earlier time the cube had a different position, namely $\Delta X = V \Delta t$

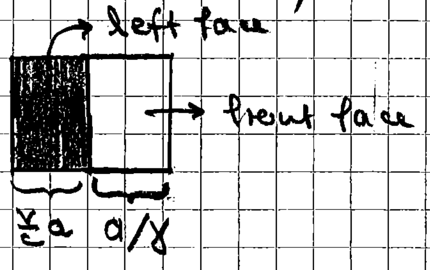
further to the left. Hence the observer sees the following picture

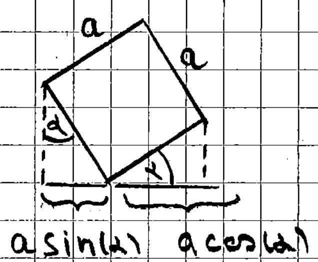


Here $\frac{v}{c} = \frac{2}{3}$ (2.2.3)

So in addition to the front face of square shape (edge-length a), the observer also sees the left face with height a and width $\Delta x = a\frac{v}{c}$. The total picture has height a and widths $a(1+\frac{v}{c})$, like a horizontal rectangle.

In Special Relativity the picture would differ insofar as the front face would be contracted in width by a factor $1/\gamma = (1-(\frac{v}{c})^2)^{1/2}$. Again for $v/c = 2/3$, it looks like





(2.2.4)

This corresponds to a rotated cube by $\sin(\alpha) = \frac{v}{c}$.

Problem 3

For a vector field

$$\vec{V} : \mathbb{R}^3 \to \mathbb{R}^3 \qquad (2.3.1)$$

and an action of $G = SO(3)$ on both $\mathbb{R}^3$ via standard (defining) representation, we have an action of $G = SO(3)$ on the set of vector fields by

1.) $$T_D \vec{V} = D \circ \vec{V} \circ D^{-1} \qquad (2.3.2)$$

Also, $G = SO(3)$ acts on the set of scalar fields $S : \mathbb{R}^3 \to \mathbb{R}$ via

$$T_D S = S \circ D^{-1} \qquad (2.3.3)$$

Note that the divergence-operator maps vector- to scalar fields,

$$\vec{\nabla} \cdot \vec{V} = S \qquad (2.3.4)$$

and the curl-operator maps vector-fields to vector fields

$$\vec{\nabla} \times \vec{V} = \vec{V}' \qquad (2.3.5)$$

The mathematical statement underlying the fact that $G = SO(3)$ acts as symmetries for equations $\vec{\nabla} \cdot \vec{V} = 0$

and $\vec{\nabla} \times \vec{V} = 0$ are now the following

$$\vec{\nabla} \cdot \left(\underbrace{T_D \vec{V}}_{\substack{\text{action of} \\ \text{vector fields}}} \right) = \underbrace{T_D}_{\substack{\text{action on} \\ \text{scalar fields}}} (\vec{\nabla} \cdot \vec{V}) \tag{2.3.6}$$

and

$$\vec{\nabla} \times \left(T_D \vec{V} \right) = T_D \left(\vec{\nabla} \times \vec{V} \right) \tag{2.3.7}$$

action on
vector fields

It is then clear that $T_D \vec{V}$ is divergence / curl - free iff that is true for $\vec{V}$ since $T_D(\vec{\nabla} \cdot \vec{V}) = 0 \Leftrightarrow \vec{\nabla} \cdot \vec{V} = 0$ and $T_D(\vec{\nabla} \times \vec{V}) = 0 \Leftrightarrow \vec{\nabla} \times \vec{V} = 0$ (the operators T_g have trivial kernel).

Let us prove (2.3.6) and (2.3.7).
Explicitly in components (2.3.2) reads

$$(T_D \vec{V})_a (X_n) = D_{ak} V_k (D^{-1}_{nm} X_m) \tag{2.3.8}$$

We write $\nabla_a := \frac{\partial}{\partial x^a}$ etc.

$$\rightarrow \nabla_a (T_D V)_a (X_n) = D_{ak} \nabla_a (V_k (D^{-1}_{nm} X_m))$$

$$= D_{ak} \nabla_b V_k (D^{-1}_{nm} X_m) D^{-1}_{ba} \tag{2.3.9}$$

$$= \underbrace{D^{-1}_{ba} D_{ak}}_{\delta_{bk}} \nabla_b V_k (D^{-1}_{nm} X_m)$$

$$= \nabla_a V_a (D^{-1}_{nm} X_m) = \vec{\nabla} \cdot \vec{V} (D^{-1} \vec{x})$$

$$= T_D (\vec{\nabla} \cdot \vec{V}) (X_n) \qquad (2.3.10)$$

which proves (2.3.6). Note that we used the chain rule:

$$\nabla_a (F \circ D^{-1}) = (\nabla_b F \circ D^{-1}) D^{-1}_{ba} \qquad (2.3.11)$$

Similarly, in components have

$$(\vec{\nabla} \times \vec{V})_a = \varepsilon_{abc} \nabla_b V_c \qquad (2.3.12)$$

hence

$$(\vec{\nabla} \times T_D \vec{V})_a = \varepsilon_{abc} \nabla_b (T_D \vec{V})_c$$

$$= \varepsilon_{abc} \nabla_b D_{ck} V_k (D^{-1}_{nm} X_m)$$

$$= \varepsilon_{abc} D_{ck} \nabla_b (V_k (D^{-1}_{nm} X_m))$$

$$= \varepsilon_{abc} D_{ck} \nabla_d V_k (D^{-1}_{nm} X_m) D^{-1}_{db}$$

$$= \varepsilon_{abc} D_{bd} D_{ck} \nabla_d V_k (D^{-1}_{nm} X_m)$$

(here we used $D^{-1}_{db} = D_{bd}$ since $D \in SO(3)$)

$$\overset{(*)}{=} D_{al} \varepsilon_{ldk} \nabla_d V_k (D^{-1}_{nm} X_m)$$

$$= D (\vec{\nabla} \times \vec{V} (D^{-1} \vec{x})) = T_D (\vec{\nabla} \times \vec{V}) (\vec{x}) \qquad (2.3.13)$$

At (*) we used

$$\varepsilon_{abc} D_{bd} D_{ck} = D_{al} \varepsilon_{ldk} \quad (2.3.14)$$

This follows from $D \in SO(3)$, since $\det D = 1$ is equivalent to invariance of ε:

$$\varepsilon_{ibc} D_{il} D_{bd} D_{ck} = \varepsilon_{ldk} \quad (2.3.15)$$

Multiplication with $D^{-1}_{la} = D_{al}$ gives

$$\varepsilon_{ibc} \underbrace{D_{il} D^{-1}_{la}}_{\delta_{ia}} D_{bd} D_{ck} = D_{al} \varepsilon_{ldk}$$

$$\Leftrightarrow \varepsilon_{abc} D_{bd} D_{ck} = D_{al} \varepsilon_{ldk} \quad \square$$

Hence (2.3.7) is also proven.

3.) A rotationally invariant vector field satisfies

$$T_D \vec{V} = \vec{V} \Leftrightarrow D \circ \vec{V} \circ D^{-1} = \vec{V}$$

$$\Leftrightarrow D \circ \vec{V} = \vec{V} \circ D \quad (2.3.16)$$

evaluate this at a point $\vec{x}$, then

$$D(\vec{V}(\vec{x})) = V(D\vec{x}) \quad (2.3.17)$$

As this must hold for all $D \in SO(3)$, it is in particular true for those D for which $D\vec{x} = \vec{x}$, i.e. where D is a rotation the axis of which is parallel $\vec{x}$. Then for these D

$$D(\vec{V}(\vec{x})) = \vec{V}(\vec{x}) \qquad (2.3.18)$$

which means that the vector $\vec{V}(\vec{x})$ is also parallel to the axis of rotation, i.e. parallel to $\vec{x}$. Hence $\vec{V}(\vec{x})$ must be of the form

$$\vec{V}(\vec{x}) = \vec{x}\ f(\vec{x}) \qquad (2.3.19)$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is a scalar function. Now,

$$D(V(\vec{x})) = V(D\vec{x})$$

is equivalent to

$$f(D\vec{x}) = f(\vec{x}) \qquad (2.3.20)$$

i.e f is $G = SO(3)$ invariant.
This means that f is constant on the orbits of $G = SO(3)$ in $\mathbb{R}^3$, i.e on the 2-spheres of constant radius r. Hence f depends on $\vec{x}$ only via $r := \|\vec{x}\|$. As a result, the most general $SO(3)$ invariant vector field $\vec{V}$ is of the form

$$\vec{V}(\vec{x}) = \vec{x}\ f(r), \quad r := \|\vec{x}\| \qquad (2.3.21)$$

A vector field of that form is divergence free iff

$$0 = \vec{\nabla} \cdot \vec{V}(\vec{x}) = \nabla_a (X_a f(r)) =$$

$$= 3 f(r) + X_a f'(r) \frac{X_a}{r}$$

$$= 3 f(r) + r f'(r) \qquad (2.3.22)$$

$$\Leftrightarrow \quad 3f + rf' = 0$$

$$\Leftrightarrow \quad \frac{f'}{f} = -\frac{3}{r}$$

$$\Leftrightarrow \quad f(r) = \frac{k}{r^3} \qquad k = \text{constant} \qquad (2.3.23)$$

The most general divergence-free spherically symmetric vector field is of Coulomb-Type

$$\vec{V}(\vec{x}) = k \frac{\vec{x}}{r^3} \qquad (2.3.24)$$

Note that this field does not exist on all of $\mathbb{R}^3$. Its domain of definition is $\mathbb{R}^3 \setminus \{\vec{0}\}$ on which it is unbounded.

A spherically symmetric vector field is always curl free:

$$(\vec{\nabla} \times \vec{V})_a = \varepsilon_{abc} \nabla_b (X_c f(r))$$

$$= \underbrace{\varepsilon_{abc}}_{\text{anti}} (\underbrace{\delta_{bc}}_{\text{sym}} f + \underbrace{X_c X_b}_{\text{in } cb} f'/r) = 0. \qquad (2.3.25)$$