Sheet 3

Problem 1

If, $\{e_a : a = 0,1,2,3\}$ is a basis of $\mathbb{R}^4$ which is orthonormal wrt to euclidean inner product

$$g(e_a, e_b) = \delta_{ab} = \mathrm{diag}(1,1,1,1) \quad (3.1.1)$$

Then a linear map $L : \mathbb{R}^4 \to \mathbb{R}^4$ is symmetric wrt. g iff

$$g(Lv, w) = g(v, Lw) \quad \forall v, w \in \mathbb{R}^4 \quad (3.1.2)$$

$$\Leftrightarrow \quad g(Le_a, e_b) = g(e_a, Le_b) \quad (3.1.3)$$

If we write

$$L(e_a) = L^c{}_a e_c \quad (3.1.4)$$

then

$$g(L^c{}_a e_c, e_b) = g(e_a, L^d{}_b e_d)$$

$$\Leftrightarrow \quad L^c{}_a g_{cb} = g_{ad} L^d{}_b \quad (3.1.5)$$

$$\Leftrightarrow \quad L_{ab} = L_{ba} \quad (3.1.6)$$

where

$$L_{ab} := g_{ac} L^c{}_b \quad (3.17)$$

In our case $g_{ac} = \delta_{ac}$ so that the map L is symmetric wrt. g iff the matrix $L^a{}_b = L_{ab}$ is symmetric (Attention: This is only true if the matrix refers to an orthonormal basis.)

$$B(\beta\vec{e}_x) = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.1.8}$$

Is obviously symmetric and refers to the standard (= orthonormal) basis.

The eigenvalues follow from the characteristic equation

$$\det\left(B(\beta\vec{e}_x) - \lambda E_4 \right) = 0$$

$$\Leftrightarrow (1-\lambda)^2 \left[(\gamma-\lambda)^2 - \beta^2\gamma^2 \right] = 0$$

$$\Leftrightarrow (1-\lambda)^2 \left[\lambda^2 - 2\gamma\lambda + 1 \right] \tag{3.1.9}$$

(having used $\gamma^2(1-\beta^2) = 1$)

$$\Leftrightarrow \lambda_{1,2} = \gamma \pm (\gamma^2 - 1)^{1/2}$$

$$= (1-\beta^2)^{-1/2} \pm \left(\frac{1}{1-\beta^2} - 1 \right)^{1/2}$$

$$= \frac{1 \pm \beta}{\sqrt{1-\beta^2}} = \frac{1 \pm \beta}{\sqrt{(1+\beta)(1-\beta)}}$$

$$= \left(\frac{1 \pm \beta}{1 \mp \beta} \right)^{1/2} \tag{3.1.10}$$

$$\lambda_{3,4} = 1 \tag{3.1.11}$$

The eigenvectors $E_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ (3.1.12)

corresponding to $\lambda_{3,4}$ are obvious.
Therefore $\lambda_{1,2}$ must lie in Span $\{e_0, e_1\}$ and are

$$E_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad E_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{3.1.13}$$

since

$$B(\beta \vec{e}_x) E_1 = \frac{1}{\sqrt{2}} \gamma (1+\beta) \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \left(\frac{1+\beta}{1-\beta} \right)^{1/2} E_1 \tag{3.1.14}$$

$$B(\beta e_x) E_2 = \frac{1}{\sqrt{2}} \gamma (1-\beta) \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \left(\frac{1-\beta}{1+\beta} \right)^{1/2} E_2 \tag{3.1.15}$$

Since $\beta < 1$ all eigenvalues are real and positive $\Rightarrow B(\beta e_x)$ is a symmetric positive definite map from $\mathbb{R}^4$ to itself.

We note that $B(\beta \vec{e}_x)$ can be written as

$$B(\beta \vec{e}_x) = \gamma\, e_0 \otimes e_0^T + \beta\gamma (e_0 \otimes e_1^T + e_1 \otimes e_0^T)$$

$$+ (\gamma - 1)\, e_1 \otimes e_1^T + E_3 \tag{3.1.16}$$

Hence

$$R(D)\, B(\beta \vec{e}_x)\, R(D^{-1}) =$$

$$\gamma\, e_0 \otimes e_0 + \beta\gamma\,(e_0 \otimes e_1^T R(D^{-1}) + R(D) e_1 \otimes e_0)$$

$$+ (\gamma - 1)\, R(D) e_1 \otimes e_1^T R(D^{-1}) + E_3 \quad (3.1.17)$$

Now,

$$R(D)\, e_1 = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 \\ \vec{e}_x \end{pmatrix} = \begin{pmatrix} 0 \\ D\vec{e}_x \end{pmatrix}$$

$$e_1^T R(D^{-1}) = (0 \;\; \vec{e}_1^{\,T}) \begin{pmatrix} 1 & 0 \\ 0 & D^T \end{pmatrix}$$

$$= (0 \;\; \vec{e}_1^{\,T} D^T) = (0 \;\; (D\vec{e}_1)^T) \quad (3.1.18)$$

Hence

$$R(D)\, B(\beta\vec{e}_x)\, R(D^{-1})$$

$$= B(\beta \vec{n}), \quad \vec{n} = D\vec{e}_x$$

$$= \gamma\, e_0 \otimes e_0 + \beta\gamma\,(e_0 \otimes n + n \otimes e_0)$$

$$+ E_3 + (\gamma - 1)\, n \otimes n^T \quad (3.1.19)$$

where $n = \begin{pmatrix} 0 \\ \vec{n} \end{pmatrix}$

or

$$B(\beta\vec{n}) = \begin{pmatrix} \gamma & \gamma\beta\vec{n}^T \\ \gamma\beta\vec{n} & E_3 + (\gamma-1)\vec{n}\otimes\vec{n}^T \end{pmatrix}$$

or

$$\left.\begin{aligned} ct &\longmapsto \gamma\,(ct + \gamma\beta\, \vec{n}\cdot\vec{x}) \\ \vec{x} &\longmapsto \vec{x} + (\gamma-1)\vec{n}(\vec{n}\cdot\vec{x}) + \beta\gamma\, \vec{n}\, ct \end{aligned}\right\} \quad (3.1.20)$$

3)

Part 3 of this exercise has already been done in the lecture:

$$\vec{X}_{\parallel} \to \gamma(\vec{X}_{\parallel} + \vec{\beta} ct) \qquad (3.1.21a)$$

$$\vec{X}_{\perp} \to \vec{X}_{\perp} \qquad (3.1.21b)$$

$$\leadsto \vec{X}' = \vec{X}_{\parallel}' + \vec{X}_{\perp}' = \gamma(\vec{X}_{\parallel} + \beta \vec{n} ct) + X_{\perp}$$

$$= \vec{X}_{\parallel} + \vec{X}_{\perp} + (\gamma - 1)\vec{X}_{\parallel} + \beta\gamma \vec{n} ct$$

$$= \vec{X} + (\gamma - 1)\vec{n}(\vec{n}\cdot\vec{X}) + \beta\gamma\vec{n} ct \qquad (3.1.22)$$

4)

The boost transformation $B(\beta\vec{n})$ leaves the plane orthogonal to e_0 and $\vec{n}$ pointwise invariant, i.e. vectors of the form $(0\ \vec{V}^T)$, with $\vec{V}\cdot\vec{n} = 0$, and transforms the plane $\mathrm{Span}\{e_0, n\}$ into itself.

Let $\vec{n} = \vec{e}_x$ (as at the beginning) then the orbit of a point (ct, x) under boosts with varying velocity β is

$$\left.\begin{aligned} ct(\beta) &= \gamma(ct + \beta x) \\ x(\beta) &= \gamma(x + \beta ct) \end{aligned}\right\} (3.1.23)$$

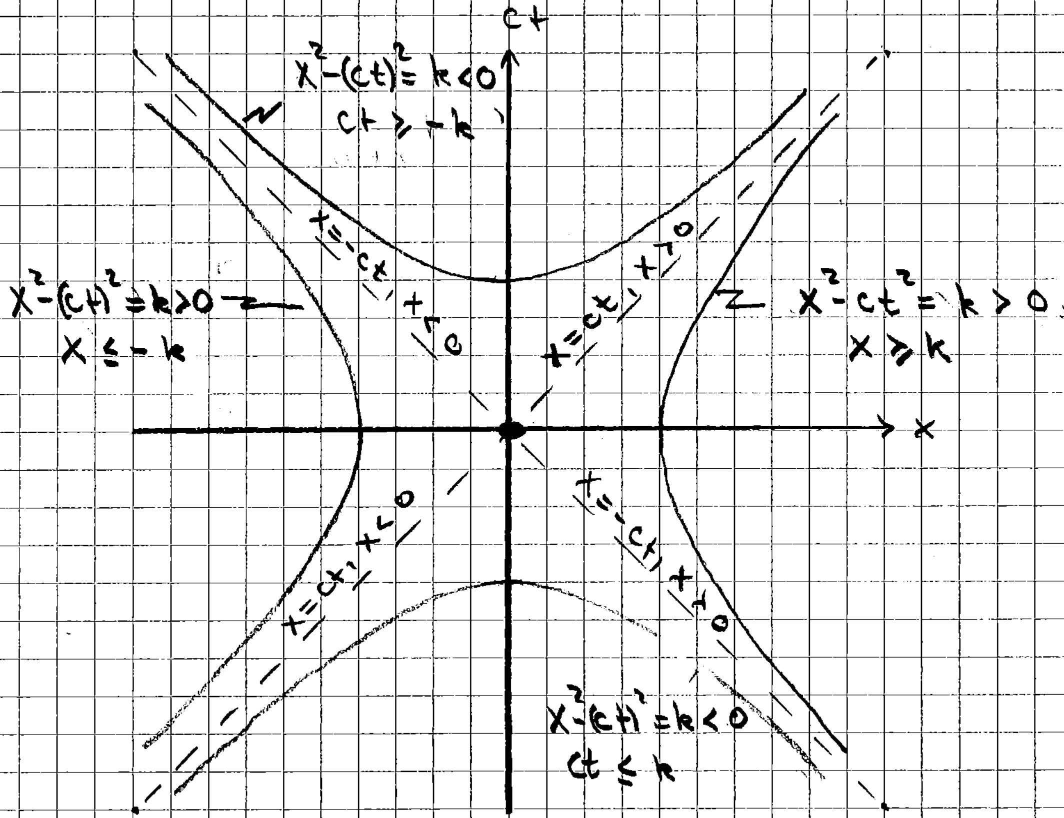
which satisfy

$$(ct(\beta))^2 - (x(\beta))^2 = (ct)^2 - x^2 \qquad (3.1.24)$$

Showing that the orbits

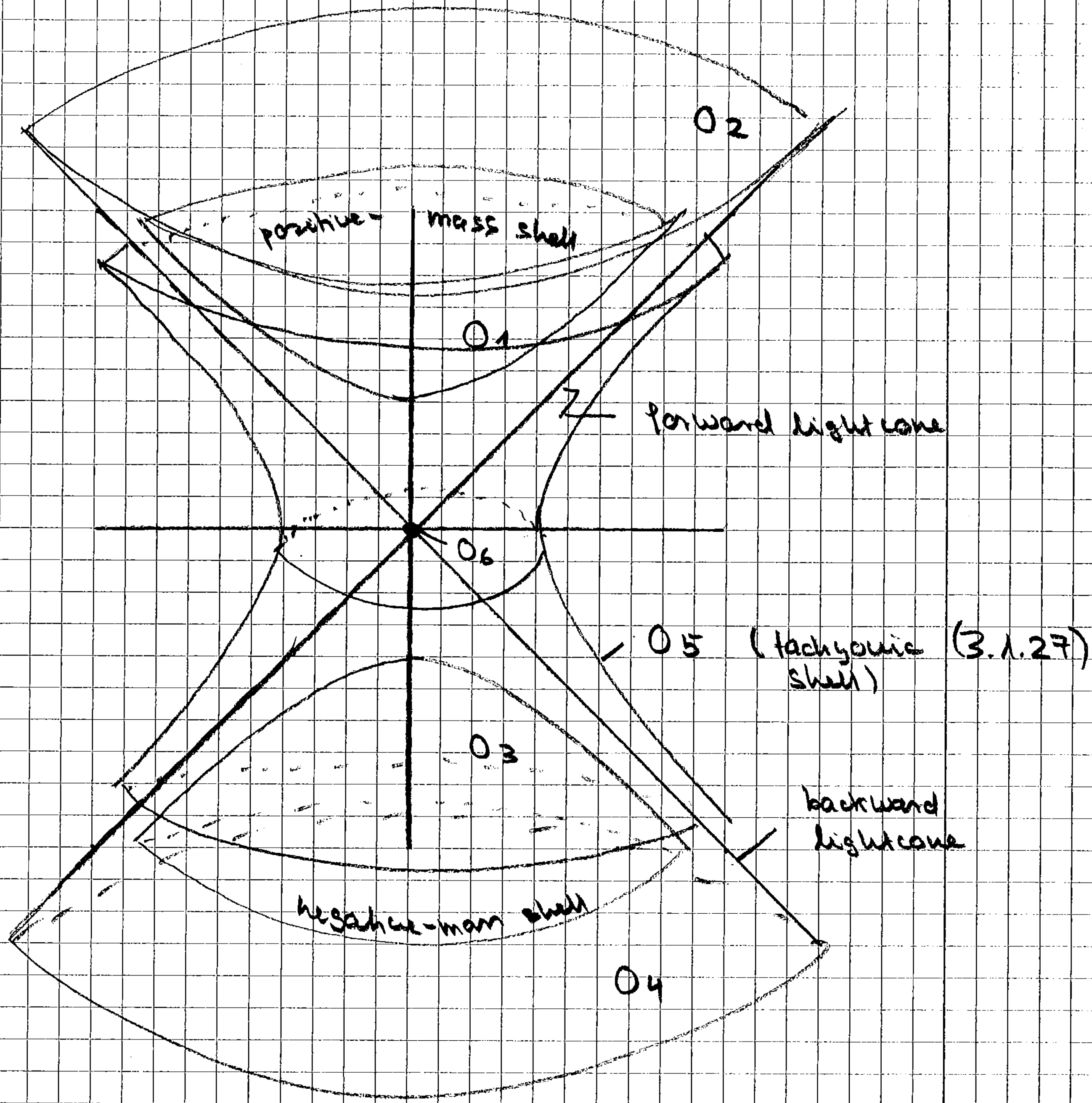
$$\mathcal{O}(ct, x) = \{((ct(\beta), x(\beta)) : \beta \in (-1,1)\} \quad (3.1.25)$$

are hyperbolae, light cones, and the origin:



(3.1.26)

Hence it appears that there are $4+4+1 = 9$ orbit types, which is, in fact, true in 1+1-dimensions. However, in $(1+n)$-dimensions where $n \geq 2$, there are only 6 since the right- and left light-cones and hyperbolae in the upper diagram are connected by spatial rotations



$$O_1 = \{ (ct, \vec{x}) : (ct)^2 - \vec{x}^2 = k > 0, \; ct > k \} \tag{3.1.29a}$$

$$O_2 = \{ (ct, \vec{x}) : (ct)^2 - \vec{x}^2 = 0, \; ct > 0 \} \tag{3.1.29b}$$

$$O_3 = \{ (ct, \vec{x}) : (ct)^2 - \vec{x}^2 > k > 0, \; ct < -k \} \tag{3.1.29c}$$

$$O_4 = \{ (ct, \vec{x}) : (ct)^2 - \vec{x}^2 = 0, \; ct < 0 \} \tag{3.1.29d}$$

$$O_5 = \{ (ct, \vec{x}) : (ct)^2 - \vec{x}^2 = k < 0 \} \tag{3.1.29e}$$

$$O_5 = \{ (0, \vec{0}) \} \tag{3.1.29f}$$

Problem 2

$$M \in GL(\mathbb{R}^n) \tag{3.2.1}$$

Endow $\mathbb{R}^n$ with euclidean scalar product

$$g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \tag{3.2.2}$$

With respect to g we can define M^T via

$$\left.\begin{aligned} g(V, MW) &= g(M^T V, W) \\ &\forall\, V, W \in \mathbb{R}^n \end{aligned}\right\} \tag{3.2.3}$$

Then MM^T is symmetric and positive definite:

$$\begin{aligned} g(V, MM^T W) &= g(M^T V, M^T W) \\ &= g(MM^T V, W), \quad \text{since } (M^T)^T = M; \end{aligned} \tag{3.2.4}$$

also

$$\begin{aligned} g(V, MM^T V) &= g(M^T V, M^T V) \\ &= \| M^T V \|^2 \geq 0 \quad \text{and} = 0 \text{ iff} \\ &M^T V = 0 \Leftrightarrow V = 0 \quad \text{as } M^T \in GL(\mathbb{R}^n). \end{aligned} \tag{3.2.5}$$

As MM^T is symmetric w.r.t. g there exists a basis with respect to which

$$g(e_a, e_b) = \delta_{ab} \quad \text{(orthonormal)} \tag{3.2.6}$$

and

$$MM^T(e_a) = \lambda_a e_a, \quad \lambda_a > 0. \tag{3.2.7}$$

Now we define the linear map $P := \sqrt{MM^T}$ by

$$P : e_a \to \sqrt{\lambda_a}\, e_a \qquad (3.2.8)$$

which is symmetric and positive definite. We call it the positive square-root of MM^T.

Let
$$O := P^{-1} M \qquad (3.2.9)$$

then
$$O^T = M^T (P^{-1})^T = M^T P^{-1} \qquad (3.2.10)$$

and
$$O O^T = P^{-1} M M^T P^{-1}$$
$$= P^{-1} P^2 P^{-1} = \mathbb{1} \qquad (3.2.11)$$

$\Rightarrow$ O is orthogonal

Hence
$$M = \underbrace{P}_{\substack{\text{pos. def} \\ \text{symm}}} \; \underbrace{O}_{\text{orthogonal}} \qquad (3.2.12)$$

showing existence. To show uniqueness assume

$$M = P_1 O_1 = P_2 O_2 \qquad (3.2.13)$$

$\Rightarrow P_1^{-1} P_2 = O_1 O_2^T =$ orthogonal

$\Rightarrow (P_1^{-1} P_2)^T = (P_1^{-1} P_2)^{-1} \Leftrightarrow P_2 P_1^{-1} = P_2^{-1} P_1$

$\Leftrightarrow P_1^2 = P_2^2 \Leftrightarrow P_1 = P_2$ (positivity) $\Rightarrow O_1 = O_2$. $\square$