Sheet 4 : Solutions

Problem 1

If the vehicle's velocity $\beta = v/c$ is large enough so that, seen from the rest frame K of the "laboratory", its contracted length $\gamma^{-1} L$ is less than the gap length l, i.e.

$$\gamma^{-1} L < l \qquad \text{("no contact condition")} \tag{4.1.1}$$

$$\Leftrightarrow \quad \gamma > L/l \tag{4.1.2}$$

$$\Leftrightarrow \quad \gamma^{-2} = 1-\beta^2 < l^2/L^2 \tag{4.1.3}$$

$$\Leftrightarrow \quad \beta > \left[1-(l/L)^2\right]^{1/2} \tag{4.1.4}$$

then, seen from the rest system K' of the vehicle, the gap is of contracted length $\gamma^{-1} l$ and the time the circuit is closed is

$$\tau = \frac{L - \gamma^{-1} l}{v} \tag{4.1.5}$$

We compare this "closure time" with the time a light signal needs (at least) to travel from one end of the vehicle to the other (wheel-to-wheel), which is

$$\tau_* := \frac{L}{c} \tag{4.1.6}$$

We ask: How does τ compare to τ_*? We would say that there is no current flowing if the contacts are closed for a time that is shorter than the time for light to travel from one end of the contact to the other, i.e. if

$$\tau < \tau_* \tag{4.1.7}$$

$$\Leftrightarrow \quad \frac{L - \gamma^{-1} l}{v} < \frac{L}{c}$$

$$\Leftrightarrow \quad L(1-\beta) < \gamma^{-1} l$$

$$\Leftrightarrow \quad L < \frac{\sqrt{1-\beta^2}}{1-\beta}\, l = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} l \tag{4.1.8}$$

("no-current condition")

But we have

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} < \left(\frac{1+\beta}{1-\beta}\right)^{1/2} \tag{4.1.9}$$

Proof: Square both sides

$$\frac{1}{1-\beta^2} < \frac{1+\beta}{1-\beta} \Leftrightarrow \frac{1}{1+\beta} < 1+\beta$$

$$\Leftrightarrow (1+\beta) > 1 \tag{4.1.10}$$

which is true for all $\beta \in (0,1)$.

Hence

$$L < \gamma l < \left(\frac{1+\beta}{1-\beta}\right)^{1/2} l \qquad (4.1.11)$$

which means that the "no contact condition" (4.1.1) in K implies the "no current condition" (4.1.8) in K'.

We conclude that if $L < \gamma l$ no photons will be emitted by the bulb.

Note that the "no current condition" (4.1.8) can also be obtained from K. Here the time the circuit is closed is

$$\bar{\tau} := (\gamma^{-1} L - l) / v \qquad (4.1.12)$$

So even if this is positive (i.e. smaller velocities than those for (4.1.1)) K will conclude that the time for a current to flow is too small if $\bar{\tau}$ is smaller than the time light needs to travel the gap l:

$$\bar{\tau} < \frac{l}{c} \iff \gamma^{-1} L - l < \beta l$$

$$\iff L < \gamma (1+\beta) l = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} l \qquad (4.1.13)$$

which is just (4.1.8).

Problem 2

1.)

Addition-law for hyperbolic tanh:

$$\tanh(\alpha+\beta) = \frac{\tanh(\alpha)+\tanh(\beta)}{1+\tanh(\alpha)\cdot\tanh(\beta)}$$

So if in

$$v_2 = \frac{v+v_1}{1+\frac{v v_1}{c^2}}$$

we set

$$\beta = \frac{v}{c} = \tanh(\alpha)$$

$$\beta_1 = \frac{v_1}{c} = \tanh(\alpha_1)$$

$$\beta_2 = \frac{v_2}{c} = \tanh(\alpha_2)$$

then it is equivalent to

$$\tanh(\alpha_2) = \frac{\tanh(\alpha)+\tanh(\alpha_1)}{1+\tanh(\alpha)\tanh(\alpha_1)}$$

$$= \tanh(\alpha_1+\alpha_2)$$

$$\Leftrightarrow \quad \alpha_2 = \alpha+\alpha_1$$

2.)

$$v_2 = \frac{\frac{c}{n} + v}{1 + \frac{cv}{nc^2}} = \frac{c}{n} \frac{1 + n\frac{v}{c}}{1 + \frac{v}{nc}}$$

$$= \frac{c}{n} \left(1 + n\frac{v}{c}\right)\left(1 - \frac{v}{nc} + O\left(\frac{v^2}{c^2}\right)\right)$$

$$= \frac{c}{n} \left(1 + n\frac{v}{c} - \frac{v}{nc} + O\left(\frac{v^2}{c^2}\right)\right)$$

$$= \frac{c}{n} + v\varphi + O\left(\frac{v^2}{c^2}\right)$$

where $\varphi = (1 - n^{-2})$

3.) If n is frequency or wavelengths dependent, then $v_1 = c/n$ is the velocity of light measured in K', and hence it is to be evaluated at the wavelength λ' measured in K'.

$$n = n(\lambda')$$

Since v_2 is the velocity measured in K, we wish to reexpress all terms involving λ' in terms of λ. The relation between λ' and λ is - to leading order, just the ordinary "non-relativistic" (i.e. linear in v/c)

Doppler formula

$$\lambda' \cong \lambda\left(1+\frac{v}{v_1}\right) = \lambda\left(1+\frac{vn}{c}\right)$$

In this formula it does not matter whether we take n to be at λ' or λ, since that would make a difference of order $\frac{v}{c}$ for n and hence of order $(v/c)^2$ for $(\lambda' - \lambda)$.

Now

$$\frac{1}{n(\lambda')} = \frac{1}{n(\lambda)} + (\lambda' - \lambda)\frac{d}{d\lambda}\left(\frac{1}{n(\lambda)}\right) + \cdots$$

$$= \frac{1}{n(\lambda)} - \lambda\frac{vn}{c}\frac{1}{n^2}\frac{dn}{d\lambda} + \cdots$$

$$= \frac{1}{n} - \frac{v}{c}\frac{\lambda}{n}\frac{dn}{d\lambda} + \cdots$$

where all n and $dn/d\lambda$ are evaluated at λ. Inserting this into

$$V_2 = \frac{c}{n(\lambda')} + v\left(1 - n^{-2}(\lambda')\right)$$

this makes a leading-order contribution only in the first term:

$$\frac{c}{n(\lambda')} = \frac{c}{n} - v\frac{\lambda}{n}\frac{dn}{d\lambda}\Big|_{\text{ev. at } \lambda}$$

Hence

$$V_2 \cong \frac{c}{n} + v\varphi$$

with $\varphi = \left(1 - n^{-2} - \frac{\lambda}{n}\frac{dn}{d\lambda}\right)$

This formula was already derived by H. A. Lorentz in 1895 (!), ten years before SR, on the basis of his aether ideas. Here it follows easily on purely kinematical considerations without any further model-assumptions.

Problem 3

$$\vec{\beta}_2 = \vec{\beta} * \vec{\beta}_1 = \frac{\vec{\beta} + \vec{\beta}_1^{\parallel} + \gamma^{-1} \vec{\beta}_1^{\perp}}{1 + \vec{\beta}\cdot\vec{\beta}_1}$$

1.)

$$\vec{\beta}_1^{\parallel} = \vec{\beta}_1 - \vec{\beta}_1^{\perp}$$

$$\curvearrowright \quad \vec{\beta}_2 = \frac{\vec{\beta} + \vec{\beta}_1}{1 + \vec{\beta}\cdot\vec{\beta}_1} - (1 - \gamma^{-1}) \frac{\vec{\beta}_1^{\perp}}{1 + \vec{\beta}\cdot\vec{\beta}_1}$$

If $\vec{\beta} = \beta \vec{n}$, then

$$\vec{\beta}_1^{\perp} = - \vec{n} \times (\vec{n} \times \vec{\beta}_1)$$

$$= - \frac{1}{\beta^2} \vec{\beta} \times (\vec{\beta} \times \vec{\beta}_1)$$

Have $\gamma^{-2} = 1 - \beta^2 \curvearrowright \beta^{-2} = (1 - \gamma^{-2})^{-1}$

$$\Rightarrow \quad \vec{\beta}_1^{\perp} = - \frac{\vec{\beta} \times (\vec{\beta} \times \vec{\beta}_1)}{1 - \gamma^{-2}}$$

Using

$$\frac{1 - \gamma^{-1}}{1 - \gamma^{-2}} = \frac{1}{1 + \gamma^{-1}} = \frac{\gamma}{1 + \gamma}$$

we get

$$\vec{\beta}_2 = \frac{\vec{\beta} + \vec{\beta}_1}{1 + \vec{\beta}\cdot\vec{\beta}_1} + \frac{\gamma}{1+\gamma} \frac{\vec{\beta} \times (\vec{\beta} \times \vec{\beta}_1)}{1 + \vec{\beta}\cdot\vec{\beta}_1}$$

Alternatively, we set

$$\vec{\beta}_1^{\perp} = \vec{\beta}_1 - \vec{\beta}_1^{\parallel}$$

and write

$$\vec{\beta}_2 = \frac{\vec{\beta} + \gamma^{-1}\vec{\beta}_1}{1+\vec{\beta}\cdot\vec{\beta}_1} + (1-\gamma^{-1})\frac{\vec{\beta}_1^{\parallel}}{1+\vec{\beta}\cdot\vec{\beta}_1}$$

Now, again with $\vec{\beta} = \beta\vec{n}$ and

$$\vec{\beta}_1^{\parallel} = \vec{n}(\vec{n}\vec{\beta}_1) = \frac{1}{\beta^2}\vec{\beta}(\vec{\beta}\cdot\vec{\beta}_1)$$

$$= \frac{\vec{\beta}(\vec{\beta}\cdot\vec{\beta}_1)}{1-\gamma^{-2}}$$

get

$$\vec{\beta}_2 = \frac{\vec{\beta} + \gamma^{-1}\vec{\beta}_1}{1+\vec{\beta}\cdot\vec{\beta}_1} + \frac{\gamma}{1+\gamma}\,\frac{\vec{\beta}(\vec{\beta}\cdot\vec{\beta}_1)}{1+\vec{\beta}\cdot\vec{\beta}_1}$$

2)

From $$\vec{\beta}_2 = \frac{\vec{\beta} + \vec{\beta}_1^{\parallel} + \gamma^{-1} \vec{\beta}_1^{\perp}}{1 + \vec{\beta}\cdot\vec{\beta}_1}$$

we get by squaring, since $\vec{\beta} + \vec{\beta}_1^{\parallel}$ is orthogonal to $\vec{\beta}_1^{\perp}$

$$\beta_2^2 = \frac{(\vec{\beta} + \vec{\beta}_1^{\parallel})^2 + \gamma^{-2}(\vec{\beta}_1^{\perp})^2}{(1 + \vec{\beta}\cdot\vec{\beta}_1)^2}$$

$$= \frac{\beta^2 + 2\vec{\beta}\cdot\vec{\beta}_1 + (\vec{\beta}_1^{\parallel})^2 + \gamma^{-2}(\vec{\beta}_1^2 - (\vec{\beta}_1^{\parallel})^2)}{(1 + \vec{\beta}\cdot\vec{\beta}_1)^2}$$

$$= \frac{\beta^2 + \gamma^{-2}\beta_1^2 + (1-\gamma^{-2})(\vec{\beta}_1^{\parallel})^2 + 2\vec{\beta}\cdot\vec{\beta}_1}{(1 + \vec{\beta}\cdot\vec{\beta}_1)^2}$$

Now,

$$(\vec{\beta}_1^{\parallel})^2 = \left[\frac{1}{\beta^2}\vec{\beta}(\vec{\beta}\cdot\vec{\beta}_1)\right]^2 = \frac{(\vec{\beta}\cdot\vec{\beta}_1)^2}{\beta^2}$$

$$= \frac{(\beta\cdot\beta_1)^2}{1-\gamma^{-2}}$$

hence

$$\beta_2^2 = \frac{\beta^2 + \gamma^{-2}\beta_1^2 + (\vec{\beta}\cdot\vec{\beta}_1)^2 + 2\vec{\beta}\cdot\vec{\beta}_1}{(1 + \vec{\beta}\cdot\vec{\beta}_1)^2}$$

Next we compute $\gamma_2^{-2} = 1 - \beta_2^2$:

$$\gamma_2^{-2} = 1 - \beta_2^2$$

$$= \frac{1 + \cancel{2(\vec{\beta}\cdot\vec{\beta}_1)} + \cancel{(\vec{\beta}\cdot\vec{\beta}_1)^2} - \beta^2 - \gamma^{-2}\beta_1^2 - \cancel{(\vec{\beta}\cdot\vec{\beta}_1)^2} - \cancel{2\vec{\beta}\cdot\vec{\beta}_1}}{(1 + \vec{\beta}\cdot\vec{\beta}_1)^2}$$

$$= \frac{1 - \beta^2 - \gamma^{-2}\beta_1^2}{(1 + \vec{\beta}\cdot\vec{\beta}_1)^2} = \frac{\gamma^{-2} - \gamma^{-2}\beta_1^2}{(1 + \vec{\beta}\cdot\vec{\beta}_1)^2}$$

$$= \frac{\gamma^{-2}\gamma_1^{-2}}{(1 + \vec{\beta}\cdot\vec{\beta}_1)^2}$$

$$\Rightarrow \quad \gamma_2 = \gamma\gamma_1 (1 + \vec{\beta}\cdot\vec{\beta}_1)$$

It follows that for

$$\|\vec{\beta}_1\| < 1, \ \|\vec{\beta}\| < 1 \Leftrightarrow \gamma, \gamma_1 < \infty$$

that $\gamma_2 < \infty \Leftrightarrow \|\vec{\beta}_2\| < 1$.

3)
$$\left.\begin{array}{l}\vec{\beta} = \beta\,\vec{n} \\ \vec{\beta}_1 = \beta\,\vec{m}\end{array}\right\}\quad \vec{n}\cdot\vec{m} = 0$$

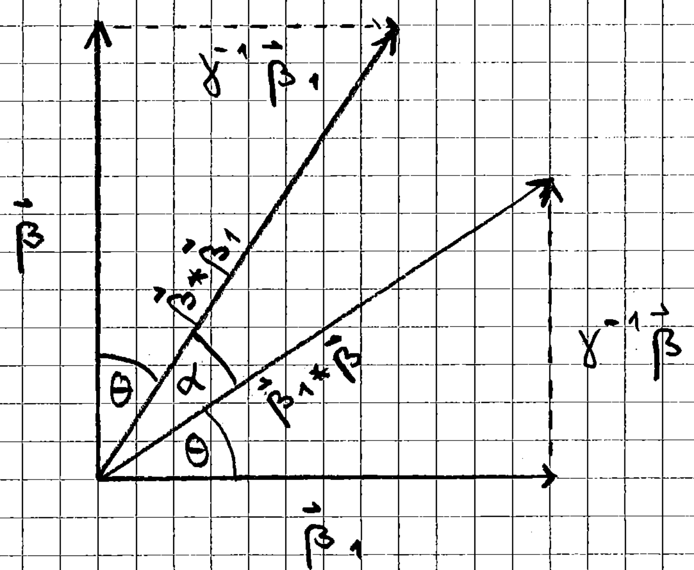
$$\vec{\beta}_2 = \vec{\beta}_1 * \vec{\beta}_2 = \beta\left(\vec{n} + \gamma^{-1}\,\vec{m}\right)$$

$$\vec{\beta}'_2 = \vec{\beta}_2 * \vec{\beta}_1 = \beta\left(\vec{m} + \gamma^{-1}\,\vec{n}\right)$$

$$\vec{\beta}_2^{\,2} = \vec{\beta}_2'^{\,2} = \beta^2\left(1 + \gamma^{-2}\right)$$
$$= \beta^2\left(2 - \beta^2\right)$$

$$\cos(\alpha) = \frac{\vec{\beta}_2\cdot\vec{\beta}'_2}{\vec{\beta}_2^{\,2}} = \frac{2\,\beta^2\,\gamma^{-1}}{\beta^2\; 1+\gamma^{-2}}$$

$$= \frac{2\,\gamma}{1+\gamma^2}$$



Drawing for $\gamma = \frac{2}{3}$

From the drawing we read off

$$\tan(\theta) = \frac{\| \gamma^{-1} \vec{\beta} \|}{\| \vec{\beta}_1 \|} = \gamma^{-1}$$

$$\alpha = 90^\circ - 2\theta$$

$$\rightsquigarrow \cos(\alpha) = \cos(90^\circ - 2\theta) = \sin(2\theta)$$

$$= 2 \sin(\theta) \cos(\theta)$$

$$= 2 \cdot \frac{\tan(\theta)}{\sqrt{1+\tan^2(\theta)}} \cdot \frac{1}{\sqrt{1+\tan^2(\theta)}}$$

$$= 2 \frac{\gamma^{-1}}{1+\gamma^{-2}} = \frac{2\gamma}{1+\gamma^2}$$

as already derived algebraically above

Problem 4

The straight line through $r \in \mathbb{R}^4$ in the timelike direction v (i.e. $v^2 > 0$) is given by

$$G_r(v) := \{ X(\lambda) = r + \lambda v \, : \, \lambda \in \mathbb{R} \} \tag{4.4.1}$$

Let $p \notin G_r(v)$ and

$$\mathcal{L}_p := \{ X \in \mathbb{R}^4 : (X-p)^2 = 0 \} \tag{4.4.2}$$

the double light-cone with vertex p.

1.) We wish to show

$$G_r(v) \cap \mathcal{L}_p = \{ q_+, q_- \} \tag{4.4.3}$$

Now, $X(\lambda) \in \mathcal{L}_p \Leftrightarrow$

$$(X(\lambda) - p)^2 = 0$$

$$\Leftrightarrow \quad (r + \lambda v - p)^2 = 0$$

$$\Leftrightarrow \quad \lambda^2 v^2 + 2\lambda \, v \cdot (r-p) + (r-p)^2 = 0$$

$$\Leftrightarrow \quad \lambda^2 + 2\lambda \frac{v}{v^2} \cdot (r-p) + (r-p)^2 / v^2 = 0 \tag{4.4.4}$$

since v timelike, hence $v^2 > 0$.

$\Rightarrow$

$$\lambda_{1,2} = -\frac{V\cdot(r-p)}{V^2} \pm \left[\frac{(V\cdot(r-p))^2}{(V^2)^2} - \frac{(r-p)^2}{V^2}\right]^{1/2}$$

$$= \frac{1}{\|V\|_\eta} \left\{-\hat{V}\cdot(r-p) \pm \left[(\hat{V}\cdot(r-p))^2 - (r-p)^2\right]^{1/2}\right\} \quad (4.4.5)$$

where
$$\hat{V} = \frac{V}{\|V\|_\eta} \quad (4.4.6a)$$

and
$$\|V\|_\eta = \sqrt{V^2} \quad (4.4.6b)$$

(4.4.5) shows that only the product $\lambda \|V\|_\eta$ matters and that we could have assumed $\|V\|_\eta = 1$, i.e. $V^2 = 1$ from the start without loss of generality. Now, in order to show the existence of two <u>real</u> solutions for (4.4.5) we must show

$$[\hat{V}\cdot(r-p)]^2 > (r-p)^2 \quad (4.4.7)$$

This is a consequence of the following

<u>Lemma</u>: Let $V \in \mathbb{R}^4$ a timelike vector and $u \in \mathbb{R}^4$ any other vector, then

$$(V\cdot u)^2 \geq V^2 u^2 \quad (4.4.8)$$

(reversed Cauchy-Schwarz inequality)

Equality in (4.4.8) holds iff u is a multiple of v.

The proof follows from the simple fact that a non-zero vector w that is orthogonal (with respect to inner product η) to the timelike vector v is spacelike, i.e. $w^2 = \eta(w,w) < 0$

Proof of Lemma: We decompose u into components parallel and orthogonal to v.

$$u = u_{\parallel} + u_{\perp} \tag{4.4.9}$$

where
$$u_{\parallel} := \frac{v\,(v\cdot u)}{v^2} \tag{4.4.10a}$$

$$u_{\perp} := u - u_{\parallel}. \tag{4.4.10b}$$

Then
$$(u_{\perp})^2 \leq 0 \quad , = 0 \text{ iff } u_{\perp} = 0 \tag{4.4.11}$$

and $(u_{\parallel})^2 = (v\cdot u)^2 / v^2$, or equivalently

$$v^2 (u_{\parallel})^2 = (v\cdot u)^2 \tag{4.4.12}$$

Hence
$$\begin{aligned} v^2 u^2 &= v^2 (u_{\parallel}^2 + u_{\perp}^2) \\ &= (v\cdot u)^2 + v^2 u_{\perp}^2 \\ &\leq (v\cdot u)^2 \end{aligned} \tag{4.4.13}$$

Since $v^2 > 0$, $u_{\perp} \leq 0$, so that $v^2 u_{\perp}^2 \leq 0$, with equality iff $u_{\perp} = 0$ ■

We apply the Lemma to our case where

$$V = \hat{v} \tag{4.4.14a}$$

$$u = r - p \tag{4.4.14b}$$

Since we required $p \notin G_+(V)$ the vector $(r-p)$ cannot be parallel to V. Hence the strict inequality (4.4.8) holds and we have

$$[\hat{v}\cdot(r-p)]^2 > \hat{v}^2(r-p)^2 = (r-p)^2 \tag{4.4.15}$$

showing that (4.4.5) has two real roots.

2)

$$\left.\begin{aligned}(q_+-p) &= (q-p)+(q_+-q)\\ (q_--p) &= (q-p)+(q_--q)\end{aligned}\right\}\ \text{are both lightlike} \tag{4.4.16}$$

and

$$(q_+-q) = \lambda\,(q-q_-)\ ,\ \lambda\in\mathbb{R}_+ \tag{4.4.17}$$

The lightlike condition is

$$0 = (q_+-p)^2 = (q-p)^2 + 2(q-p)(q_+-q) + (q_+-q)^2$$

$$0 = (q_--p)^2 = (q-p)^2 + 2(q-p)(q_--q) + (q_--q)^2$$

$$\Rightarrow\ -(q-p)^2 = 2(q-p)(q_+-q) + (q_+-q)^2 \tag{4.4.18a}$$

$$= 2(q-p)(q_--q) + (q_--q)^2 \tag{4.4.18b}$$

If we multiply (4.4.18b) with λ and add it to (4.4.18a), then, because of (4.4.17) the terms $\sim 2(q-p)$ cancel and we get

$$-(1+\lambda)(q-p)^2 = (q_+ - q)^2 + \lambda(q_- - q)^2$$

$$(4.4.17) \longrightarrow = \lambda^2 (q_- - q)^2 + \lambda (q_- - q)^2$$

$$= \lambda(1+\lambda)(q_- - q)^2 \qquad (4.4.19)$$

and since $\lambda \in \mathbb{R}_+$, so that $(1+\lambda) \neq 0$,

$$-(q-p)^2 = \lambda(q_- - q)^2 = \lambda(q - q_-)^2$$

$$= (q_+ - q)\cdot(q - q_-) \qquad (4.4.20)$$

Note that $(q_+ - q)$ and $(q - q_-)$ are timelike and parallel, so that

$$(q_+ - q)\cdot(q - q_-) = \|q_+ - q\|_\eta \, \|q_- - q\|_\eta \qquad (4.4.21)$$

whereas $(q-p)$ is spacelike, so that

$$(q-p) = -\|q-p\|_\eta^2 = -\|p-q\|_\eta^2 \qquad (4.4.22)$$

Hence

$$\|p-q\|_\eta^2 = \|q_+ - q\|_\eta \cdot \|q - q_-\|_\eta \qquad (4.4.23)$$

This is Robb's Theorem (1914). It holds for all q between q_+ and q_- !

3) We subtract (4.4.18b) from (4.4.18a) and get

$$0 = 2(q-p)[(q_+-p)-(q_--p)]$$
$$+(q_+-q)^2-(q_--q)^2$$

$$= 2(q-p)\cdot(q_+-q_-)$$
$$+(q_+-q)^2-(q_--q)^2 \qquad (4.4.24)$$

Hence

$$(q-p)\cdot v = 0 \iff (q-p)(q_+-q_-)=0$$
$$\iff (q_+-q)^2 = (q_--q)^2$$
$$\iff \|q_+-q\|_2 = \|q-q_-\|_2 \,. \qquad (4.4.25)$$

4)
$$G_r(v) = \{ r+\lambda v : \lambda \in \mathbb{R}\}, \qquad (4.4.26a)$$

$$G_{r'}(v') = \{ r'+\lambda' v' : \lambda' \in \mathbb{R}\}. \qquad (4.4.26b)$$

We wish to find pair $(\lambda, \lambda') \in \mathbb{R}\times\mathbb{R}$ such that the connecting vector

$$X'(\lambda') - X(\lambda) = (r'+\lambda' v') - (r+\lambda v) \qquad (4.4.27)$$

is perpendicular to v and v', that is, to the timelike plane span $\{v, v'\}$. This leads to a system of two linear equations for two variables λ and λ':

$$(r' + \lambda' v')\cdot v - (r + \lambda v)\cdot v = 0 \qquad (4.4.28a)$$
$$(r' + \lambda' v')\cdot v' - (r + \lambda v)\cdot v' = 0 \qquad (4.4.28b)$$

$$\Leftrightarrow \quad \lambda'(v\cdot v') - \lambda v^2 = (r - r')\cdot v$$
$$\lambda' v'^2 - \lambda(v\cdot v') = (r - r')\cdot v'$$

$$\Rightarrow \quad \underbrace{\begin{pmatrix} (v\cdot v') & -v^2 \\ v'^2 & -(v\cdot v') \end{pmatrix}}_{A} \begin{pmatrix} \lambda' \\ \lambda \end{pmatrix} = \begin{pmatrix} (r - r')\cdot v \\ (r - r')\cdot v' \end{pmatrix} \qquad (4.4.29)$$

$$\rightsquigarrow \quad \det(A) = -(v\cdot v')^2 + v^2 v^2 \qquad (4.4.30)$$

By the reversed Cauchy-Schwarz inequality (4.4.8) [which does apply here because v and v' are timelike], we have $\det(A) < 0$. The inequality is strict because we assumed v and v' to be not parallel. If the lines intersect they both lie in the plane $\mathrm{Span}\{v, v'\}$ and the only vector $x'(\lambda') - x(\lambda)$ orthogonal to that plane is the null vector. In that case the unique solution (λ', λ) to (4.4.29) is such that $X'(\lambda') - X(\lambda) = 0$, i.e only the intersection point on the two straight lines are "simultaneous to each other". If the two lines are skew, $(X'(\lambda') - X(\lambda)) \neq 0$ and is orthogonal to $\text{span}\{V, V'\}$